THE ITERATIVE DEFERRED ACCEPTANCE MECHANISM

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Abstract. The last few years have seen an increase in the use of sequential online mechanisms, instead of the traditional direct counterparts, in college admissions in many countries, including Germany, Brazil, and China. We describe these mechanisms and identify their shortcomings in terms of incentives and outcome properties. We introduce a new family of mechanisms for one-sided matching markets, which improve upon these shortcomings. Unlike most mechanisms in the literature, which ask students for a full preference ranking over all colleges, our mechanisms instead ask students to sequentially make choices or submit partial rankings from sets of colleges. These are used to produce a tentative allocation at each step. If at some point it is determined that a student can no longer be accepted into a college, then she is asked to make another choice among those schools that would tentatively accept her. Participants following the simple strategy of choosing the most preferred college in each step is a robust equilibrium that yields the Student-Optimal Stable Matching.

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1. Introduction

The field of market design has developed rapidly in recent years, both in terms of the range of objectives that are studied—different notions of efficiency, stability, fairness, etc.—and also in the number of applications and their evaluations, in both the field and the lab. In the typical framework, the attainability of a given objective is evaluated in terms of mechanisms that require the relevant agents submit preferences over sets of outcomes before a clearinghouse combines them using some predetermined criteria to produce an allocation. This induces a game in which the action space of the participants consists of rankings over their outcomes. By studying the incentive properties of these games, one can then see how equilibrium outcomes relate to the objectives of the market designer. These mechanisms, therefore, have two common properties: they are direct (in the sense that the participants are asked for their relevant types, in this case their preferences) and induce a simultaneous move game: all agents simultaneously interact only once.

There are many theoretical and practical reasons for focusing on direct mechanisms. First, the revelation principle guarantees that nothing is lost by using direct mechanisms instead of alternative action spaces. Second, in the induced games, the participants have a simple strategy space, whereas strategies in sequential games may consist of large sets of contingency plans over information structures. Finally, if the mechanism is strategy-proof, it is very simple for a participant, with truth-telling being the expected behavior.

In this paper, we consider centralized college admissions, where the designer is interested in implementing stable outcomes. The Gale-Shapley student-proposing deferred acceptance procedure (DA) is often deemed the ideal theoretical solution for this problem due to its incentive and fairness properties [Balinski and Sönmez, 1999]. However, despite the availability of DA, the last few years have seen the emergence of "iterative" (or sequential) mechanisms for matching students to schools and colleges, sometimes on a very large scale. Prominent examples include the college admission mechanism for the Chinese province of Inner Mongolia [Chen and Pereyra, 2016, Gong and Liang, 2016], used for matching more than 200,000 students to universities per year, the mechanism used in Brazil to determine matching for more than two million prospective public university students per year, and the mechanism currently being used in German university admissions [Grenet et al., 2019].[1] In these iterative mechanisms, the designer and the participants potentially interact multiple times, and between these interactions, some information about intermediate outcomes is communicated to the students. In the first part of the paper, we analyze the iterative mechanisms used in Brazil, Germany, and the province of Inner Mongolia in China. We show that these mechanisms have undesirable properties, such as the inability to provide reliable information about where students could be accepted and susceptibility to a new type of manipulation, denoted by manipulation via cutoffs. Manipulation via cutoffs are situations in which groups of students with high exam grades temporarily inflate the cutoff grades at some colleges and change their options in the last step, with the objective of reducing the competition faced by specific low-grade students. We show that, due to the specific characteristics of college admissions in these countries, these manipulations are feasible both in Brazil and Inner Mongolia and provide anecdotal evidence that they take place in real life in the latter case. Importantly, these undesirable incentive properties distort the stability of the final outcome.

An obvious solution to all these problems would be a switch to DA. However, designers might be opposed to a sharp change in the system or value the multiple interactions with students and the possibility of intermediate communication. Finally, the designer might prefer the iterative feature of the current mechanism. In this paper, we try to answer the following question: Can a designer implement a stable allocation, maintaining the iterative nature of the system with the possibility of intermediate communication with students? And if so, what is the "cost" of the use of the resulting procedure when compared to DA?

In the second part of the paper, we propose a family of iterative mechanisms for implementing stable outcomes in many-to-one matching markets. In the mechanisms we propose, instead of requesting a full preference over all private outcomes from each participant, they

[1] Another example of the current use of iterative mechanisms is the school district in Wake County, NC [Dur et al., 2018].

are instead repeatedly asked to choose one option or to submit a (potentially partial) ranking over the options available. These choices are used to produce tentative allocations. Some information about these allocations may be given back to the participants before asking them for further choices and/or rankings until a final allocation is produced. These mechanisms, therefore, resemble a sequential and flexible implementation of DA, in which instead of asking for preference rankings and using them in an algorithm, the agents themselves are repeatedly asked to make choices from sets of available options. This family of mechanisms is denoted by the Generalized Iterative Deferred Acceptance Mechanisms (GIDAM). We also consider a prominent special case of GIDAM for exam-based college admissions, where the students choose one university at a time: the Iterative Deferred Acceptance Mechanism (IDAM).

The GIDAM mechanisms share some properties with the standard direct implementation of DA, but not all. While DA is strategy-proof, GIDAM mechanisms may lack even a weakly dominant strategy. On the other hand, all students following the simple strategy of choosing the most preferred option at each step (denoted by *straightforward strategy*) constitute a robust equilibrium: at each step, regardless of the choices made before, choosing the most preferred option—or submitting truthful partial rankings when allowed—from that point on first-order stochastically dominates any deviating strategy, for any distribution of beliefs (Theorem 1). This result comes from the fact that deviating strategies are indistinguishable, from an observer's perspective, from a "truthful" behavior for some different preferences. The outcome produced in that equilibrium is the same as the one produced by DA: the student-optimal stable matching (Proposition 1). Thus, the designer who has a preference for iterative allocation procedures can reach a desirable outcome. However, instead of implementing in dominant strategies, she can use an ordinal perfect Bayesian equilibrium implementation in straightforward strategies.

So far, we have been agnostic about the potential benefits of iterative mechanisms and instead took the iterative implementation as an external preference of the designer. Lately, however, there is growing evidence that these procedures might have benefits not captured by the standard school choice model.

Recent experimental evidence shows that stable allocations are reached significantly more often under iterative versions of DA than under direct DA [Klijn et al., 2019, Bó and Hakimov, 2020]. This result is driven by higher rates of truthful behavior under the iterative mechanism than under DA.[2]

Another reason why iterative stable mechanisms may be desirable is that the steps involved in the production of the final allocation are more transparent. In standard DA, the rankings students submit are used in a sequential computer-operated process, which may not be entirely understood by the participants. Making the agents themselves make choices and see their effects, each step of the iterative mechanisms is simpler and better understood. In France, for instance, DA was abandoned in favor of a sequential procedure in 2018, following complaints about the lack of transparency of DA.[3]

Finally, by construction, the iterative mechanisms allow for the designer to reveal intermediate information to the students between the steps of an iterative procedure. In the mechanisms used in Inner Mongolia and Brazil, the emphasis of the designer is on publishing intermediate cutoff grades that should help students update their expectations about their chances of acceptance at a university.[4]

The numerous interactions between participants and the designer involved in implementing a GIDAM would seem infeasible 15 years ago. The use of these procedures has been made possible, in practice, by the Internet, which allows students to easily interact multiple times

[2]This evidence is closely related to growing experimental and empirical evidence that the strategic simplicity of DA may not be matched by an understanding, on the part of the agents, of its incentives [Chen and Sönmez, 2006, Pais and Pintér, 2008, Ding and Schotter, 2019, Rees-Jones, 2018, Hassidim et al., 2020, Chen and Pereyra, 2015]. For an extended survey of the experimental literature of school choice see Hakimov and Kübler [2020].

[3]Hakimov and Raghavan [2020] formalize the transparency for centralized allocations, and indeed iterative mechanisms that accept the submission of one object at a time are more transparent than the direct mechanism according to their notion.

[4]Grenet et al. [2019], Hakimov et al. [2020] emphasize the usefulness of this communication if students have to acquire costly information about their preferences over universities, and this acquisition is possible between the steps of the procedure.

with a central clearinghouse via a website or even a mobile application. Still, in principle, the procedure might take a long time before the stable allocation is reached. In the third part of this paper, we show that the comparison between DA and IDAM presents a trade-off between the length of submitted preferences in DA and the number of steps that IDAM takes to produce its outcome. Section 5 evaluates this trade-off through theoretical and simulation results. We run simulations comparing the number of steps it takes for the IDAM mechanism to produce an outcome and the minimum length of a rank-ordered list necessary for truth-telling to be an equilibrium in DA. These show that the relative advantage of the IDAM is higher when the ratio of students to seats is higher. Interestingly, when the number of students equals the number of seats, the simulations also show that IDAM produces an outcome in fewer steps in scenarios where DA needs longer rank-ordered lists and vice versa.

Proofs absent from the main text and additional details can be found in the Appendix.

Related literature. This paper mainly relates to two lines of research in market design. One is the family of works that evaluate, from both a positive and a normative perspective, mechanisms used in the field in college admissions and school choice. While evaluating the college admission process in Turkey, Balinski and Sönmez [1999] showed that the Gale-Shapley student-proposing deferred acceptance procedure (DA) [Gale and Shapley, 1962] is characterized as the "best" fair mechanism, in that it is strategy-proof and Pareto dominates any other fair mechanism (that is, it is constrained efficient). In fact, variations of the DA mechanism are used in many real-life student matching programs around the world. Examples include the college and secondary school admissions in Hungary [Biró, 2011], high school admissions in Chicago [Pathak and Sönmez, 2013] and New York City [Abdulkadiroğlu et al., 2009], and elementary schools in Boston [Abdulkadiroglu et al., 2006]. Other mechanisms, such as the college-proposing DA, the so-called "Boston mechanism," and the "Parallel mechanism" are used to match millions of students to schools and colleges worldwide [Chen and Kesten, 2017, Abdulkadiroğlu and Sönmez, 2003, Balinski and Sönmez, 1999]. Gong and Liang [2016] apply the mechanism currently in use to match students to universities in Inner

Mongolia. Grenet et al. [2019] analyze the system used for college admissions in Germany, which combines a sequential phase and a direct revelation phase. In subsection 5.2, we show that one instance of the GIDAM mechanism also presents this combination and has good practical properties.

This paper is also related to the study of sequential mechanisms. Kagel et al. [1987] show that, although the second-price auction is isomorphic to an English auction, experiments show that behavior is significantly different when comparing both, with truthful behavior more prevalent in the latter. Ausubel [2004] and Ausubel [2006] propose sequential auction mechanisms for multiple (homogeneous and heterogeneous, respectively) objects. While there are direct mechanisms that implement the same outcomes in dominant strategies, the author argues that the proposed sequential mechanisms are simpler and preserve the participants' privacy.

In a recent paper, Li [2017] provides a theoretical justification for why some sequential mechanisms perform better than their direct counterparts. That justification is based on a refinement of strategy-proofness, denoted obvious strategy-proofness (OSP), in which the realization that a certain strategy is dominant does not rely on contingent reasoning. The author shows that a family of mechanisms, which includes the English auction, is OSP, therefore providing a a theoretical explanation for the results in Kagel et al. [1987].[5] When it comes to stable mechanisms, however, Ashlagi and Gonczarowski [2018] show that no OSP mechanism yields stable matchings.[6]

Experimentally comparing the behavior under DA and the IDAM mechanism in the school choice setup, Bó and Hakimov [2020] show that the truthful equilibrium in IDAM, which produces the student-optimal stable matching, predicts behavior better than the dominant

[5]Baccara et al. [2012] evaluate the use of an obviously strategy-proof mechanism — Sequential serial dictatorship — in the assignment of offices for the faculty in a new building, and analyze the effect of faculty network on the choices in the mechanism. The coordination of choices would be harder to execute using a direct mechanism. Thus, one potential benefit of using iterative mechanisms might come from the possibility of accommodating non-standard preferences.

[6]The authors show, however, that there is an OSP stable mechanism when the preferences on one side of the market satisfy an acyclicity condition. This is a very restrictive condition, which is not satisfied, for example, by the college admission process in Brazil.

strategy in DA, also leading to a larger proportion of stable outcomes. Klijn et al. [2019] also present evidence in that direction. Echenique et al. [2016] analyze the behavior under an iterative DA mechanism in the two-sided matching setup and show that most realized stable outcomes were receiver-optimal. Similarly, Kagel and Levin [2009] show experimental evidence that subjects more often behave in line with the equilibrium prediction in the sequential mechanisms in Ausubel [2004] than with the dominant strategy direct counterpart. These results indicate that the behavior more consistent with the equilibrium prediction in these sequential implementations is not entirely captured by the refinement proposed in Li [2017].

Other papers have evaluated non-direct iterative mechanisms for matching students to colleges or schools. Dur et al. [2018] use the fact that the school choice mechanism used in the Wake County Public School System allows for students to interact multiple times with the procedure as a method for empirically identifying strategic players. Interestingly, the dynamic nature of the procedure, and the information made available to the participants during the process, makes it somewhat comparable to the IDAM mechanism. Dur and Kesten [2019] and Haeringer and Iehlé [2019] evaluate the sequential use of direct mechanisms for a single allocation problem. In Dur and Kesten [2019], assignments made at each step are final, and in Haeringer and Iehlé [2019] students might reject their matchings and submit a potentially updated preference in the following round.

A rich series of papers also consider sequential mechanisms that implement stable matchings in equilibrium, including Alcalde and Romero-Medina [2000], Alcalde and Romero-Medina [2005], Romero-Medina and Triossi [2014], and Klaus and Klijn [2017]. While many of these mechanisms implement stable allocations in equilibrium, the determination of equilibrium strategies depends on coordination between students in a way that is significantly more demanding than the equilibrium strategy that IDAM has, which depends solely on (partial) information about the student's own preferences over colleges.

2. Iterative mechanisms in the field

In this section, we provide a brief description of three iterative mechanisms that are currently in use for college admissions. These are: (i) the mechanism used for a large portion of the public universities in Brazil, which we will denote as the *Brazilian Mechanism*, (ii) the mechanism used for some public universities in Germany, denoted as the *German Mechanism*, and (iii) the mechanism used for universities for residents of the Chinese province of Inner Mongolia, denoted as the *Inner Mongolia Mechanism*.

Even though these procedures have many differences, for the most part, they share the same basic setup, which we will use for their descriptions.

There is a set of students $S = \{s_1, \ldots, s_n\}$, and of colleges $C = \{c_1, \ldots, c_m\}$ with fixed capacities (a maximum number of students who can be matched to them) $(q_{c_1}, \ldots, q_{c_m})$. Colleges rank all students based on some score (which in Brazil and Inner Mongolia come from a national exam, and in Germany from a combination of different criteria). If using a national exam, different colleges[7] may use different weights for the various parts of the exam. For example, economics programs could give a higher weight to the exam's math section, while medical programs could give a higher weight to the biology section. Denote by $z_c(s)$ student s's resulting exam grade in college c. Colleges may also have a minimum acceptance grade, representing the minimum value of $z_c(s)$ a student s must have to be acceptable at c, denoted by $\underline{z}_c$.

Given a set of students applying to a certain college, one commonly used information is the *cutoff* grade for that college. A cutoff grade represents the lowest grade necessary to be accepted at a college, given the set of students applying to it. When looking at all colleges' cutoff values, a student can thus infer which ones would accept her if all other students' choices remain constant. Before the centralized iterative mechanisms were introduced in Brazil and China, it was common for students to see the historical values of the cutoffs for

[7]In the countries considered, as in many others, students apply directly to specific programs in the colleges or universities. For simplicity, though, we refer only to "colleges" whenever the distinction is not necessary.

the different colleges as an indication of where they should apply, given their information about their own exam grades or ability. One of the advantages of the new procedures would be to allow the students to make that assessment in "real-time" instead of only based on historical data.

2.1. **The German Mechanism.** The procedure that we denote by German Mechanism is originally called DoSV, *Dialogorientiertes Serviceverfahren*. Our description of the procedure and its characteristics is drawn from Grenet et al. [2019]. The German Mechanism is used for admissions to some competitive university programs across the country and was introduced in 2012. In the winter term of 2015/16, more than 180,000 students applied to 465 programs in 89 universities.

The German Mechanism operates in three stages:[8]

- **Stage 1**: During this period, students submit a ranked ordered list with at most 12 colleges to the central clearinghouse.
- **Stage 2** (32 days):
  - During this period, each college c submits the students' scores z_c. When the scores from a college c are received, the clearinghouse automatically sends emails with offers to the q_c highest-scoring students with respect to z_c, who had c among the colleges in their list submitted during Stage 1.
  - A student with one or more offers can choose to accept one of them and leave the procedure, or can choose to hold some or all of these offers.
  - When some student rejects an offer made by some college c, the clearinghouse sends another offer to the student with the highest score in z_c, who had c among the colleges in their list submitted during Stage 1 but had not yet received an offer from c.

[8]While the description we provide omits some details of the procedure, it provides the key elements necessary for our analysis and for identifying shortcomings that are also present in the actual procedure. A detailed description of the other details can be found in Grenet et al. [2019].

- **Stage 3**: The seats that were not taken by the students who left in Stage 2—including the offers that were held but not accepted— and the students who did not leave during Stage 2 are matched using the Gale-Shapley college-proposing deferred acceptance mechanism [Gale and Shapley, 1962], using the ranked ordered list submitted by students in Stage 1 as their preferences and the scores submitted by the colleges in Stage 2 to rank the students. Students who rejected an offer from a college in Stage 2 are removed from that college's ranking, however.

Therefore, the German Mechanism matches students to colleges first through a dynamic offers procedure (Stage 2) and then uses a standard (constrained list) college-proposing deferred acceptance for the remaining seats. Notice that offers from colleges that are held but not accepted by the end of Stage 2 are treated as rejected offers before going to Stage 3.

2.2. **The Brazilian Mechanism.** In the period between 2010 and 2016, the precise rules which define the Brazilian Mechanism were changed multiple times. The version that we describe, due to its simplicity, is the one used in the year 2010. Although later versions have different modifications, to the best of our knowledge, all the problems we identify are also present in the later versions to the best of our knowledge.

The mechanism runs for four days.

- During each day $t = \{1, 2, 3, 4\}$, students may each choose a college to apply to. If a student makes no choice, her last choice is used again, if any. At the end of each of the first three days, the following is executed for each college c:
  - If the number of students who chose c and have an exam grade at that college higher than $\underline{z}_c$ is smaller than q_c, the cutoff grade ζ_c^t is set to $\underline{z}_c$.
  - Otherwise, the cutoff grade ζ_c^t is set to be the q_c^{th} highest grade at that college among those who chose it on that day.
- The values of $\zeta_{c_1}^t, \ldots, \zeta_{c_m}^t$ are made public.
- At the end of the fourth day, a student-college matching is produced, as follows:

- For each college c, the top q_c students who have an exam grade higher than $\underline{z}_c$ and chose c on the last day are matched to it.
- All students who were not among the ones above remain unmatched.
- Final cutoffs, calculated in the same way, are made public.

2.3. **The Inner Mongolia Mechanism.** The Inner Mongolia Mechanism is used to match residents of the province of Inner Mongolia in China to seats reserved for those students in universities across the country. Our description of the procedure is drawn from Gong and Liang [2016]. In the Inner Mongolia mechanism, all colleges use the same grades in a national exam, that is, for all c, c', $z_c = z_{c'} = z$. The students are partitioned into k tiers $S = S_1 \cup S_2 \cup \cdots \cup S_k$, where the grades of all those in S_1 are greater than those in S_2, and so on. That is, if $s \in S_i$, $s' \in S_j$, $i < j$ implies that $z(s) > z(s')$. The number of tiers and the number of students in each tier varies from year to year, but the former ranges from eight to 11.

When the procedure starts, as in the Brazilian Mechanism, students can choose to apply to one of the colleges available. While students make these choices, each college's cutoff grades are calculated and made public in real-time. As in the Brazilian Mechanism, cutoff grades represent the lowest grade in z necessary to be accepted into each college, given the choices that all students made. Students can revise their choice as many times as they want, and the cutoff values are updated continuously accordingly.

After a pre-determined T number of minutes,[9] the matchings of the students in S_1 are finalized. If by the end of this time a student chose a certain college and the cutoff grade at that college is lower than her exam score, then the student is accepted into the college she chose. If, on the other hand, her last choice after T minutes is a college with a cutoff grade above her score, she will be left unmatched.[10]

[9]To the best of our knowledge, the value of T is either 60 or 180 minutes.

[10]Students who are left unmatched during the Inner Mongolia Mechanism participate in a scramble procedure, which allocates leftover seats. We do not model this phase.

After the matchings for students in S_1 terminate, the remaining students again have T minutes to make choices, observe cutoffs, and revise their choices, involving the seats not taken by the students in S_1. As for the previous tier of students, after T minutes, the matches of the students in S_2 are finalized. Some will be matched to the last college they chose, and some will be left unmatched. This procedure keeps on going until all k tiers of students are finalized.

3. Shortcomings of the current mechanisms

In this section, we present some shortcomings that we identified in the mechanisms described in the previous section. Since the Brazilian and the Inner Mongolia mechanisms share more similarities with themselves than with the German Mechanism, the nature of the shortcomings also shares that relation.

To explain the results, we need to make an assumption and provide a few definitions. The assumption is that students have **strict preferences** over the set of colleges and remaining unmatched. That is, they are not indifferent between any two colleges and may prefer to be left unmatched than to be matched to some colleges. Next, we say that a student s **justifiably envies** a student s' if s is matched to college c, s' is matched to c', s prefers c' to c, and $z_{c'}(s) > z_{c'}(s')$. Finally, a matching is **wasteful** if there is a college c which is left with an empty seat and a student who would rather be matched to c than the match she is left with.[11]

We start with the German Mechanism. One problem of the German Mechanism is that accepting offers in Stage 2 may hurt students.

Remark 1. In the German Mechanism, if a student accepts an offer during Stage 2, she may justifiably envy another student at the final allocation, or the final allocation may be wasteful.

[11]Also, if that college has a minimum grade, that student has a grade above it.

The reason for that is clear. If a student s accepts an offer during Stage 2, it is possible that another offer from a more-preferred college c^* could arrive later in that stage or during the algorithmic offers process that takes place in Stage 3. In this case, c^* would go down their ranking of students and either make an offer to another student s', in which case s would justifiably envy s', or c^* could end up leaving an empty seat, in which case the matching would be wasteful. This is not a purely theoretical possibility. In fact, Grenet et al. [2019] show empirically that the simple fact that an offer is sent to a student during Stage 2 significantly increases the likelihood that it will be accepted, despite these not being made by more desirable universities.[12]

The second problem that we identify with the German Mechanism comes from the fact that it consists of a sequentialized version of the college-proposing deferred acceptance mechanism [Gale and Shapley, 1962], and as a result has some of the incentive shortcomings of the standard implementation of that mechanism [Roth, 1982], as shown below.

Remark 2. In the German Mechanism, students may obtain better outcomes by strategically rejecting offers during Stage 2, and/or submitting ranked ordered lists that do not represent their true preferences.

One problem with the shortcoming above is that, by improving the outcomes of students who "strategically manipulate" their choices and reports, the German Mechanism may induce an advantage to students who engage in these manipulations and/or have access to information that could assist these manipulations.

Next, let us consider the Brazilian and Inner Mongolia mechanisms. A common characteristic between both mechanisms is that the choices made by the students before these are used to produce the allocation (i.e., those made during the first three days in the Brazilian Mechanism, and those made between the finalization of the matches of each tier) have no

[12]The authors of that paper interpret these acceptances as resulting from an endogenous formation of preferences with regret avoidance.

direct effect on the outcomes. That is, as long as the final choices, used when the matchings are produced, remain the same, the outcome will also not be changed.

Since cutoffs are calculated and made public during these "practice runs" that are present in both mechanisms, they could inform students about whether an application to a certain college is likely or not to be accepted in the end. If they do, then even though they do not have a direct effect on the final outcomes, they could provide information that guides students to better outcomes. However, the problem is that since students are not restricted by which college they can apply to and when, these cutoff values can freely fluctuate up and down, without necessarily providing any reliable information about which colleges would or would not accept a student at the end of the process.

Remark 3. During the "practice runs" of the Brazilian and Inner Mongolia mechanisms, each college's cutoff values may go up or down from one day to the next.

Since the cutoff values may fluctuate, for a given college, a student who has a grade higher than the cutoff cannot be sure that she will be accepted into that college if she chooses it, and a student who has a grade lower than that cutoff cannot be sure that she would not be accepted.[13]

In the next subsection, we discuss another shortcoming of the Brazilian and Inner Mongolia mechanisms, which is new to the literature, and has some indications that it may be affecting real-life outcomes.

3.1. **Manipulations via cutoffs.** A manipulation via cutoffs occurs when a group of students artificially increase the cutoff values of some college as a way to prevent other students' applications and then, shortly before matchings are finalized, vacate those seats so that students with a lower exam grade, aware of that manipulation, take their places. We define manipulability via cutoffs below.

[13]A previous working version of this paper showed, using data from 2016, that cutoff values for many programs did fluctuate substantially, both up and down, from one day to the next.

Definition 1. A mechanism is **subject to manipulations via cutoffs** if there exists a strategy profile that results in the temporary increase of the cutoff grade of a college.

Notice that the increase must be temporary, implying that at some point it is reversed. In what follows, we will say that a student presents *straightforward behavior* if she always chooses her most preferred college among those which have cutoff values below her grade. Since under the Brazilian and Inner Mongolia mechanisms students are not constrained to which colleges they can apply to and when they can change their applications, these mechanisms are subject to these strategic manipulations.

Remark 4. The Brazilian and Inner Mongolia mechanisms are subject to manipulations via cutoffs.

The example below shows how manipulations via cutoffs can happen in the case of the Brazilian Mechanism.

Example 1 (Manipulation via cutoffs)**.** Consider the set of students $S = \{s_1, s_2, s_3, s_4\}$ and of colleges $C = \{c_1, c_2, c_3\}$, each with capacity $q_i = 1$ and no minimum scores. Students' preferences are as follows:

$$P_{s_1} : c_1 \ c_2 \ c_3$$

$$P_{s_2} : c_1 \ c_2 \ c_3$$

$$P_{s_3} : c_1 \ c_2 \ c_3$$

$$P_{s_4} : c_2 \ c_1 \ c_3$$

Students' exam grades at the colleges are as follows:

	c_1	c_2	c_3
s_1	100	100	100
s_2	200	200	200
s_3	300	300	300
s_4	400	400	400

Suppose that the Brazilian Mechanism is used, and that students present straightforward behaviors. At the end of each day, the cutoff values would then be as follows (the cutoffs at $t = 4$ represent the final allocation cutoffs):

	c_1	c_2	c_3
$t = 1$	300	400	0
$t = 2, 3, 4$	300	400	200

The matching produced will therefore be μ:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 & \varnothing \\ s_3 & s_4 & s_2 & s_1 \end{pmatrix}$$

Suppose, however, that students s_1 and s_4 collude and modify their behavior, acting instead as follows:

- During $t = 1, 2, 3$, student s_1 chooses college c_3 and student s_4 chooses college c_1.
- On day $t = 4$, student s_1 chooses college c_1 and student s_4 chooses college c_2.

Assuming that the other students present straightforward behavior, the cutoff values at the end of each day would be as follows:

	c_1	c_2	c_3
$t = 1$	400	$---$	100
$t = 2$	400	300	100
$t = 3$	400	300	200
$t = 4$	100	400	200

The matching produced will be μ':

$$\mu' = \begin{pmatrix} c_1 & c_2 & c_3 & \varnothing \\ s_1 & s_4 & s_2 & s_3 \end{pmatrix}$$

Student s_1 is better off under μ' than under μ, while s_4 is matched to the same college in both cases.

In other words, successful manipulations via cutoffs consist of a set of students S^H—$S^H = \{s_4\}$ in the example above— "holding" seats for some time in colleges and "releasing" them so that a set of students S^T—$S^T = \{s_1\}$ in the example above—can take them right before their matchings are determined. In order for these manipulations to succeed, the students who are not in the coalition have to respond in a straightforward way to the last cutoff values they see. This can be considered a reasonably mild requirement. It does not require that the other students constantly follow straightforward behavior, but only that they do not choose, in the last period in which they can make a choice, a college where the cutoff value is above their grade in that college.

Other than that, the S^H coalition must be able to significantly increase the value of the cutoff when they apply to a college. This may not be easy. After all, colleges typically accept hundreds or thousands of students every year, and a coalition of hundreds of high-achieving students performing these potentially risky manipulations does not seem realistic. In many countries (including Brazil and China), however, students apply directly to specific programs at the universities, so even though the universities as a whole accept hundreds or thousands of students, the number of seats at each program is often below 100, and many times lower than 30 or 20. Moreover, even those seats are often subdivided. In China, the seats in each program are partitioned between seats reserved for candidates from specific provinces. In Brazil, federal universities partition the seats in the programs into five sets of seats, reserved for different combinations of ethnic and income characteristics [Aygün and Bó, 2013]. Finally, universities sometimes offer only a subset of the total number of seats in a program through the centralized matching process. In fact, the median number of seats offered in each option available during the January 2016 selection process in Brazil, where more than 228,000 seats in public universities were offered, was *five*.[14]

There is evidence that this type of manipulation takes place in real life. In Inner Mongolia, there is evidence that students engage in manipulations via cutoffs, as documented by China News:[15]

> "(...) in fact, since 2008, the clearinghouse found that some high scoring students applied to a college with lower cutoff score. For example, their score allows them to go to PKU or Tshinghua, but they chose Beijing Polytech first. On the other hand, some other students, from the same high school often, applied to colleges that their score would not allow them to go initially (...) [the] system shows that their rank is below the capacity—so they can't be admitted under usual terms—however they do not revise their choices."

Even more remarkably, there are indications that high schools are coordinating students' actions:

> "(...) the clearinghouse noticed that, 2 or 3 min before the deadline, the ranking of students in the system is changing—this is the evidence that high schools are organizing their own high scoring students to occupy seats for low scoring students."

Regarding the Brazilian Mechanism, while we could not find any article in the media describing manipulations via cutoff, there is at least one video on YouTube describing how to perform the manipulation.[16]

The Brazilian mechanism's shortcomings are closely related to the motivation behind the introduction of activity rules in the combinatorial clock auction [Ausubel et al., 2006, Ausubel

[14]Each program was partitioned into five options: one for each combination of characteristics related to affirmative action. Each one was treated like a college in our framework. In other words, the median number of seats in each one of these partitions was five.

[15]Source (in Chinese): http://www.chinanews.com/edu/2014/09-04/6562740.shtml. (Accessed on December 9, 2017).

[16]Source: https://www.youtube.com/watch?v=WTDxkW0cIGQ.

and Baranov, 2014]. In these auctions, participants' bids might be inconsistent over time, and agents might have the incentive to use "sniping behavior," in which bidders conceal their true intentions until the very end of the auction. This behavior is somewhat related to manipulation via cutoffs, where coalitions of students also "conceal" their true preferences until the last period. The solution proposed by the authors is to use "activity rules," which are rules that restrict participants' bids to guarantee that they satisfy consistency properties, such as some Revealed Preference axiom. In the next section, we will introduce the Generalized Iterative Deferred Acceptance Mechanisms, a family of iterative mechanisms for matching problems that include the ones dealt with by the mechanisms we evaluated. As we will show below, they improve upon the shortcomings that we identified above. For one thing, they differ from the Brazilian and Inner Mongolia mechanisms in that they restrict the participants' behavior in a way that is analogous to the activity rules used in these combinatorial auctions.

4. The Generalized Iterative Deferred Acceptance Mechanisms

In this section, we introduce the generalized version of our proposed mechanisms, denoted as the Generalized Iterative Deferred Acceptance Mechanisms (GIDAM). In this generalized version, we consider a general setup, in which the admission criteria used by colleges may be more general than one simply based on exam grades, allowing, for example, for the use of affirmative action policies or variations in financial aid.[17] This version also allows for the same students and colleges to be matched under different contractual terms, as in the matching with contracts model introduced by Hatfield and Milgrom [2005]. Finally, we also allow for cases where students might submit not only one choice at a time but also rankings over the available options.

A special case of GIDAM, which considers the same setup of college admissions based on national exam grades, is introduced in subsection 4.1.

A **matching with contracts market** is a tuple $\langle S, C, T, X, P_S, F_C \rangle$:

[17]See, for example, Hafalir et al. [2013], Aygün and Bó [2013], Shorrer and Sóvágó [2017], Hassidim et al. [2020], and Yenmez [2018].

(1) A finite set of **students** $S = \{s_1, \ldots, s_n\}$,

(2) A finite set of **colleges** $C = \{c_1, \ldots, c_m\}$,

(3) A vector of **contractual terms** $T = (t_1, \ldots, t_\ell)$,

(4) A set of **valid contracts** $X \subseteq C \times S \times T$,

(5) A list of strict **student preferences** $P_S = (P_{s_1}, \ldots, P_{s_n})$ over $X \cup \{\varnothing_s\}$[18], and the respectively derived weak preferences R_S,

(6) A list of **college choice functions** over sets of contracts $F_C = (f_{c_1}, \ldots, f_{c_m})$, where for every $c \in C$ and $I \subseteq X$, $f_c : 2^X \to 2^X$, $\{(c, s, t), (c, s', t')\} \subseteq f_c(I) \implies s \neq s'$ and $(c', s, t) \in f_c(I) \implies c' = c$.

For any $I \subset X$, $s \in S$, and $c \in C$, denote $I_s \equiv \{(c, s', t) \in I : s' = s\}$, $I_c \equiv \{(c', s, t) \in I : c' = c\}$, $s(I) \equiv \{s \in S : \exists (c, s, t) \in I\}$, and $c(I)$ to be defined analogously. We abuse notation and let $c(x)$ and $s(x)$ be the college and student in contract x, respectively. An **outcome** is a set of contracts $Y \subseteq X$ such that Y contains at most one contract per student, that is, $|Y_s| \leqslant 1$ for each $s \in S$. Denote by $\mathcal{X}$ the set of all outcomes. An outcome Y is **individually rational** if for every student s, $Y_s R_s \varnothing$ and for every college c, $Y_c = f_c(Y_c)$. Define by **maximum rank function** a function $\pi : \mathbb{Z}^+ \to \mathbb{N} \cup \{\infty\}$ that defines, for each step $t = 0, \ldots, T^{Max}$, what is the maximal length of a ranking that a student may submit.

A **GIDAM mechanism** consists of the following steps:

$t = 0$: A signal about the set of feasible allocations is broadcast.[19] Additionally, each student is given an individualized menu of contracts, consisting of the contracts involving said student that colleges deem acceptable and the null option $\varnothing$.[20] Each student who is given a non-empty menu is asked to submit an ordered list with at

[18]In some places we abuse notation and also use P_s over sets with only one contract. Here, $\varnothing$ represents the null contract for student, representing a student remaining unmatched. We also assume that a student's preference is over contracts in which she is involved and $\varnothing$.

[19]These signals consist of deterministic functions from subsets of allocations to some messages. That is, by observing those signals, students may make inferences about the set of feasible allocations at each step. A detailed and formal description of the signal function and all other aspects of the GIDAM mechanism can be found in the Appendix.

[20]That is, these contracts would be chosen if each one of them were the only option provided to the involved college.

most $\pi(0)$ contracts in their menu. After all students submit their lists (or opt not to), these are used to perform a cumulative offer process. Students who are offered a menu with contracts but opt not to submit are left unmatched. That is, students one at a time offer their highest-ranked contract on the list submitted to the involved college. The colleges choose among all contracts offered, cumulatively, with their choice functions. Whenever a contract is rejected, the student involved in it offers the next highest-ranked contract, if any. The step ends whenever every student has a contract held by a college or has all of those on the submitted list rejected.

$0 < t \leqslant T^{Max}$: At the beginning of the step, a signal about the set of allocations that are still feasible is broadcast. There are two cases: (i) If $\pi(t) = \infty$ and $T^{Max} < \infty$, every student is given an individualized menu of contracts, which consists of the contracts involving said student that colleges would accept[21]—while having all contracts that were offered in previous steps still available to these colleges— in addition to the null option $\varnothing$. (ii) If $\pi(t) \neq \infty$ or $T^{Max} = \infty$, each student who does not have a contract being held by a college is given an individualized menu of contracts, consisting of the contracts involving said student that colleges would accept—while having all contracts that were offered in previous steps still available to these colleges—in addition to the null option $\varnothing$. Each student who is given a non-empty menu is asked to submit an ordered list with at most $\pi(t)$ contracts in their menu. As in the previous step, the same cumulative offer process is undertaken, where the lists submitted by the students in this and previous steps are used to offer contracts to colleges, which are considered together with those offered in previous steps. Students who are offered a menu with contracts but opt not to submit are left unmatched.

[21]We say that a college c would accept a contract x if $x \in f_c(A \cup \{x\})$, where A is the set of contracts that were offered to c in previous steps.

The process ends after the step $t = T^{Max}$ or whenever the set of contracts held by all colleges does not change from one step to the next. Denote that last step by T^*.[22] A formal definition of the mechanism can be found in the Appendix.

Notice that menus always include the null option $\varnothing$, so whenever it helps exposition, we will omit it from our examples. The public signals do not play a role in the results we present, but in general they may affect other incentives induced by the mechanism and be useful in terms of transparency by allowing the participants to follow the procedure's intermediate steps. In subsection 4.1, for example, the public signals will be the cutoff grades at each college, as in the Brazilian and Inner Mongolia mechanisms.

Example 2. Consider a matching with contracts problem in which there are four colleges $C = \{c_1, c_2, c_3, c_4\}$, each with one seat available, and four students $S = \{s_1, s_2, s_3, s_4\}$. Colleges may accept students with or without financial aid. Colleges have a specific criterion for choosing contracts: contracts without financial aid are always preferred to contracts with aid. When comparing contracts that do not differ in that dimension, students with higher exam scores are preferred.[23] Students' grades in the national exam follow their indexes: s_1 has the highest grade, s_2 the second highest, etc. Let the maximum rank function be such that $\pi(t) = 2$ for every $t \geqslant 0$ and $T^{Max} = \infty$.

The table below shows, for each student, the menu of contracts offered in the first step (which, for all students, contain all possible contracts with colleges), and a list that is submitted by each student in the first step. We represent contracts with financial aid with the letter F and without financial aid with N. In the first list submitted by student s_1, therefore, she lists college C_1 first with financial aid followed by C_4, also with financial aid. Student s_4 opted to submit a list containing only one option, which is, of course, also a valid list.

[22]Depending on the properties of the college choice functions, it is possible to have a GIDAM mechanism with $T^{Max} = \infty$ that never terminates. However, for all the results that follow, we will make assumptions of these functions that guarantee that this will not happen.

[23]We do not argue that these preferences are typical or realistic. We use them because they are simple but allow for an informative example of the steps of the GIDAM mechanism.

The lists submitted are used in a cumulative offer process. In it, student s_1 has her top contract tentatively accepted, since no contract without financial aid is offered during that process. Moreover, although student s_3 has her contract with C_2 with financial aid rejected, her contract without financial aid is tentatively accepted, since colleges always prefer those. As a result, s_2 has both contracts in her list rejected. Student s_4's only contract is also rejected.

Step $t = 0$

Student	Tentative match	Menu offered	List submitted
s_1	$\varnothing$	C_1: F N; C_2: F N; C_3: F N; C_4: F N	C_1: F $\succ_1$ C_4: F
s_2	$\varnothing$	C_1: F N; C_2: F N; C_3: F N; C_4: F N	C_1: F $\succ_2$ C_2: F
s_3	$\varnothing$	C_1: F N; C_2: F N; C_3: F N; C_4: F N	C_2: F $\succ_3$ C_2: N
s_4	$\varnothing$	C_1: F N; C_2: F N; C_3: F N; C_4: F N	C_1: F $\succ_4$

The table below shows the menus offered in the next step and the choices we consider students to have made. Notice that the number of options in the menus offered to students s_2 and s_4 are different. Since s_4 has a low exam grade and C_2 tentatively holds a student without financial aid, contracts with C_2 would no longer be accepted. Also, while the student with the highest exam grade is tentatively matched to college c_1, a contract without financial aid is offered in the menu to student s_2.

Step $t = 1$

Student	Tentative match	Menu offered	List submitted
s_1	C_1 [F |]	None	None
s_2	$\varnothing$	C_1 [| N] C_2 [| N] C_3 [F | N] C_4 [F | N]	C_1 [| N] $\succ_2$ C_4 [F |]
s_3	C_2 [| N]	None	None
s_4	$\varnothing$	C_1 [| N] C_3 [F | N] C_4 [F | N]	C_1 [| N] $\succ_4$ C_4 [F |]

In step $t = 2$, only student s_4 will be given a menu, with only three contracts. This happens even though s_1 had her previous match to C_1 rejected. The mechanism continued down her submitted list in step $t = 0$ and matched her to C_4 with financial aid:

Step $t = 2$

Student	Tentative match	Menu offered	List submitted
s_1	C_4 [F |]	None	None
s_2	C_1 [| N]	None	None
s_3	C_2 [| N]	None	None
s_4	$\varnothing$	C_3 [F | N] C_4 [| N]	C_3 [F |] $\succ_4$ C_4 [| N]

Given the submitted list, the final matching is produced by the end of that step:

Step $t = 3$

Student	Final match
s_1	C_4 (F, —)
s_2	C_1 (—, N)
s_3	C_2 (—, N)
s_4	C_3 (F, —)

The example above highlights some of the main characteristics of the GIDAM mechanisms. Students only have to submit a list when the absence of that information would not allow the mechanism to determine where their next tentative allocation (if any) should be. Student s_1, for example, did not have to submit anything after the first step, despite the fact that her tentative match was rejected after the second step. Contracts that are no longer feasible for a student are not offered in their menus, reducing the number of options to consider. Students are also free to choose the length of the list they submit, up to the limit established by the maximum rank function π.

We can define a **matching function** μ that represents the outcome of the GIDAM mechanism, where for every $c \in C$, $\mu(c) = f_c\left(A^{T^*}(c)\right)$, and for every $x \in \mu(c)$, let $\mu(s(x)) = x$. Whenever $T^{Max} = \infty$ or there is a $t \leqslant T^{Max}$ such that $\pi(t) = \infty$, we say that the GIDAM is **unbounded**. When GIDAM is unbounded, therefore, a student is able to express, either over time or via a ranking in some steps, a sequence of choices over as many contracts as she wishes.

A **random outcome** is a probability distribution over the set of all outcomes $\mathcal{X}$. An outcome $X' \subseteq X$ is **stable** if it is individually rational and there is no college c and set of contracts $X'' \subset X$ such that $X'' \neq f_c(X')$, $X'' = f_c(X' \cup X'')$ and for every $s \in s(X'')$,

$X''_s R_s X'_s$. More specifically, we say that a student s and college c form a **blocking pair** under X' if s has a contract in $X'' \backslash X'$. A stable outcome is the **student-optimal stable allocation** if every student weakly prefers it to any other stable outcome.

In order to guarantee that the outcomes and incentives of the GIDAM mechanisms satisfy desirable properties, it is necessary to impose some restrictions on the colleges' choice functions. The first one comes from Hatfield and Kojima [2010]:

Definition 2. Contracts in X are **unilateral substitutes** for college c under f_c if there do not exist contracts $x, z \in X$ and a set of contracts $Y \subseteq X$ such that $s(z) \notin s(Y)$, $z \notin f_c(Y \cup \{z\})$, and $z \in f_c(Y \cup \{x, z\})$.

Another condition that we use comes from Aygün and Sönmez [2013]:

Definition 3. The choice function f satisfies **irrelevance of rejected contracts (IRC)** if $x \notin f(X' \cup \{x\})$ implies $f(X' \cup \{x\}) = f(X')$ for all $X' \subset X$ and $x \in X \backslash X'$.

Finally, the last property that will be used was introduced in Hatfield and Milgrom [2005]:

Definition 4. The choice function f satisfies the **law of aggregate demand** if for all $Y \subseteq Z$, $|f(Y)| \leqslant |f(Z)|$.

Lemma 1. *Assume that for every college $c \in C$, f_c satisfies IRC, and contracts in X are unilateral substitutes. Then, for every student s and $0 \leqslant t \leqslant t' \leqslant T^*$, if the set of contracts in the menu given to s in step t is non-empty, then all contracts in a menu given in step t' are also in the menu given in step t.*

Lemma 1 posits that once a contract becomes unavailable for a given student, that contract will never become available again, regardless of the strategies used by the students. This shows that one piece of information given by the mechanism after each step—the set of acceptable contracts available to the student—constitutes reliable information about the contracts that are not available anymore for a student, as opposed to the Brazilian and Inner

Mongolia mechanisms. We define a formal generalization of the "straightforward behavior" [Roth and Sotomayor, 1992] when interacting with the GIDAM mechanism:

Definition 5. A strategy of student $s \in S$ is **straightforward with respect to** P^* if for every step t in which a non-empty menu $\psi^t(s)$ is offered by the mechanism, s submits a ranking with the top $k(\psi^t(s), t)$ options in $\psi^t(s)$, ordered as in P^*, where $k : 2^X \times \mathbb{N} \to \mathbb{N}$ is a function such that, for every t, $1 \leqslant k(\cdot, t) \leqslant \pi(t)$, and $k(\cdot, t^\infty) = \left|\psi^{t^\infty}(s)\right|$, where t^∞ is the highest value of t such that $\pi(t) = \infty$.

Therefore, a strategy is straightforward in the GIDAM when, in every step, the student submits either her full preference over the contracts in the menu offered (whenever $\pi(t)$ is large enough) or some *truncation* of her true preference. Moreover, when there are multiple steps in which a student can submit an unbounded ranking over contracts, she **must** rank all the alternatives presented in a menu at some step in which that is allowed.[24] When $\pi(t) = 1$ for all t, the definition of straightforward strategy reduces to one of straightforward behavior in Roth and Sotomayor [1992]: every time the student is asked to make a choice, she picks her most preferred alternative with respect to her true preference. When students follow straightforward strategies, the outcome produced by the unbounded GIDAM is of a well-known type:

Proposition 1. *Assume that, for every college $c \in C$, f_c satisfies IRC and contracts are unilateral substitutes. If all students' strategies are straightforward with respect to P_S, there is a finite number of steps T^* after which the outcome of any unbounded GIDAM mechanism is the student-optimal stable outcome with respect to P_S.*

The proof of Proposition 1 is based on the fact, shown in Hirata and Kasuya [2014], that the cumulative offer process that takes place during the GIDAM mechanism is *order independent*

[24] Note that these include the null option $\varnothing$, and therefore contracts might be deemed unacceptable even when the ranking submitted is supposed to include all options.

and when students follow straightforward strategies the outcome is the student-optimal stable outcome. As a result, all combinations of such strategies yield the same result.[25]

When colleges' choice functions satisfy IRC, unilateral substitutes, and the law of aggregate demand, a direct mechanism that produces the student-optimal stable outcome is strategy-proof [Aygün and Sönmez, 2013, Hatfield and Kojima, 2010]. That is, submitting her true preference ranking over contracts is a weakly dominant strategy for every student. One may be tempted to conclude that this will imply that straightforward strategies, which are the equivalent of truth-telling in this dynamic setting, are also dominant under the GIDAM mechanism. However, the proposition below shows that not only is this not the case, but that the students may not have any dominant strategy at all.

Proposition 2. *A student may not have a weakly dominant strategy under the GIDAM mechanism.*

Not following a straightforward strategy may be profitable because, in contrast to the direct mechanism, an agent may influence others' actions by modifying the signals received by them. So, for example, if a student has a strategy that depends in some way on the signals produced by one's actions, or even on the sequence of menus that are presented or the timing of the rejection in a particular choice, that fact could be exploited. However, we will show that the profile in which students follow straightforward strategies constitutes a robust equilibrium. The equilibrium concept that we use is a refinement of the Perfect Bayesian Equilibrium.

Definition 6. A strategy profile together with a belief system is an **ordinal perfect Bayesian equilibrium (OPBE)** if, at every information set, every deviation from the equilibrium strategy is stochastically dominated by following it.

[25] This might come as a surprise in light of Ergin [2000], who shows that *consistency* — a property that is indirectly related to this order independence — would be incompatible with stable rules. One important difference is that in GIDAM, in every step before the last, allocations are *tentative*. Therefore, different orders in which the proposals are made do not restrict the allocations as in the cases considered in that paper.

The definition above is intentionally informal, but its formal version can be found in the Appendix. When a strategy profile is an OPBE, therefore, the probability of obtaining the most preferred contract, the two most preferred contracts, the three most preferred contracts, etc., is weakly greater when following the equilibrium strategy compared to any deviating strategy when starting from any step. Equivalently, no deviating strategy yields a better expected utility for any utility function that represents the students' ordinal preferences. For that, we consider the extensive-form game induced on the students by the GIDAM mechanism. We allow students to have uncertainty regarding other students' preferences and exam grades. The sequence of events is as follows:

(1) Step $t = 0$: Nature draws the values of X and P from a joint distribution ξ, and each student s observes the realization of X_s and P_s.
(2) Steps $1 \leqslant t \leqslant T^{Max}$: students interact with the GIDAM mechanism. That is, in each step t, every student $s \in S$ receives a menu of contracts and a maximum rank value $\pi(t)$ and has to submit a ranking over those contracts, as in the description of the mechanisms above. At the end of the step, the public signals are observed by all students. The mechanism terminates at some step $T^* \leqslant T^{Max}$.
(3) Step $T + 1$: Students are matched to their outcomes produced by the GIDAM mechanism.

Our main result shows that, when facing this game, students following straightforward strategies constitutes an OPBE.[26]

Theorem 1. *Consider a maximum rank function π and an unbounded GIDAM mechanism using it. Let ω be any belief system and σ^* be a strategy profile in which all strategies are straightforward with respect to P_S. Then σ^* together with ω is an OPBE of the game induced by the GIDAM mechanism.*

[26] Ehlers and Massó [2015] describe restrictive conditions under which strategy profiles in (static) direct stable mechanisms constitute an OBNE (the direct mechanisms counterpart of OPBE) under incomplete information. However, these conditions are not especially restrictive when we focus on the student-optimal stable mechanisms, which is strategy-proof. Our result shows that, when considering this specific selection over the set of stable matchings, that result extends to this sequential mechanism.

The proof of Theorem 1 is fundamentally based on the fact that although the space of deviating strategies is significantly large, they are all indistinguishable, from the perspective of an observer, from a student following a straightforward strategy for some different preference. This allows us to evaluate deviating strategies in all of their paths, which may include multiple interactions that the student may have with the mechanism. Without this, it would be difficult to determine the final outcome of a generally specified deviation.[27]

4.1. **Iterative Deferred Acceptance.** Here we introduce the Iterative Deferred Acceptance Mechanism (IDAM), which is an application of the GIDAM mechanisms for college admissions problems, like those tackled by the mechanisms described in section 2, where the criterion used by colleges to select candidates may be summarized by cutoff grades. An **exam-based college matching market** is a tuple $\langle S, C, q, P_S, z, \underline{Z} \rangle$:

(1) A finite set of **students** $S = \{s_1, \ldots, s_n\}$,
(2) A finite set of **colleges** $C = \{c_1, \ldots, c_m\}$,
(3) A **capacity vector** $q = (q_{c_1}, \ldots, q_{c_m})$,
(4) A list of strict **student preferences** $P_S = (P_{s_1}, \ldots, P_{s_n})$ over $C \cup \{s\}$[28],
(5) A list of **vectors of exam scores** $z = (z(s_1), \ldots, z(s_n))$, where for each $s \in S$, $z(s) = (z_{c_1}(s), \ldots, z_{c_m}(s))$, are the exam scores that student s obtained, respectively, at college $c_1 \ldots, c_m$. We assume that for every $s, s' \in S$ and $c \in C$, $z_c(s) = z_c(s') \implies s = s'$, and
(6) A list of **minimum necessary scores** $\underline{Z} = \left(\underline{z}_{c_1}, \ldots, \underline{z}_{c_m}\right)$.

The set of contracts, colleges' choice functions and public signals are derived from the above as follows:

- The set of valid contracts is $X = \{(s, c, z_c(s)) : s \in S, c \in C \text{ and } z_c(s) \geqslant \underline{z}_c\}$. That is, the valid contracts are between all colleges and the students who have an exam grade

[27] In fact, in most sequential matching mechanisms in the literature (for example, Alcalde and Romero-Medina [2005], Triossi [2009], and Romero-Medina and Triossi [2014]) the number of times an agent interacts with the mechanism is either exogenously given or is one in equilibrium.
[28] Here s represents a student remaining unmatched to any college.

at least as high as the minimum at that college. The contractual term is the exam grade itself. Since there is only one contract between each student and each college, we can refer to the menus of contracts as simply being menus of colleges. Moreover, students' preferences over contracts are directly derived from their preferences over colleges.

- For a given set of contracts Y, colleges' choice functions f_c select the top q_c contracts in Y_c with respect to z_c if $|Y_c| \geqslant q_c$, and all contracts in Y_c otherwise.
- The public signals are the colleges' cutoffs (the minimum required exam grade at that college, given the current set of contracts being held).

Finally, we let $T^{Max} = \infty$ and for all t, $\pi(t) = 1$. It is easy to see that f_c, as defined above, satisfies unilateral substitutes and the law of aggregate demand. This implies that Proposition 1 and Theorem 1 also hold for the IDAM mechanism.

Corollary 1. *If students follow straightforward strategies, the outcome of the IDAM mechanism is the student-optimal stable outcome.*

Corollary 2. *All students following straightforward strategies with respect to their true preferences constitute an OPBE of the game induced by the IDAM mechanism.*

Let ζ_c^t be the value of the cutoff of college c made public in step t, as defined above. In light of the definition of f_c, Lemma 1 leads to the following conclusion:

Corollary 3. *(Cutoff grades never go down) For every* $0 \leqslant t \leqslant T^*$ *and* $c \in C$, $\zeta_c^t \geqslant \zeta_c^{t-1}$.

As described in subsection 3.1, manipulations via cutoffs consist of temporarily inflating the cutoff value of a college and then reducing it. The corollary above implies, therefore, that manipulation via cutoffs is not feasible:

Remark 5. The IDAM mechanism is not manipulable via cutoffs.

It is also worth noting that, while the process that takes place during the execution of the IDAM mechanism resembles a "worker-proposing" version of the salary adjustment in Kelso and Crawford [1982], our results shows that, at least when students are the only strategic agents, the *process itself* is an equilibrium when interactions are restricted in the way defined by the IDAM mechanism.

5. Time feasibility

The number of steps that it takes for the GIDAM or IDAM mechanisms to produce the Student-Optimal Stable Outcome when students follow straightforward strategies depends on the interaction of multiple variables, such as students' preferences, the maximum rank function, the colleges' choice functions (or students' exam grades), etc. It is important, however, to have some sense of how many steps that will take, and if there are other viable alternatives to reduce the time necessary to produce the outcome.

First, we can make two intuitive remarks to provide a baseline for the expected range for the steps these mechanisms may take. For simplicity, we restrict our attention to IDAM mechanisms. Note that the IDAM mechanism takes the longest time among all GIDAM mechanisms, as it allows to select only one college at a time.

Remark 6. For any set of students, colleges, capacities, vectors of exam scores, and minimum necessary scores, there are preference profiles such that when students follow straightforward strategies, the IDAM mechanism produces the student-optimal stable outcome after one step.

A preference profile that would lead to the observation above can be constructed as follows. Take any preference profile P_S and, given the set of students, colleges, capacities, vectors of exam scores, and minimum necessary scores, produce a stable matching μ^*. Next, let the preference profile P'_S be the same as P_S except that for every student s who is matched to a college under μ^*, move the position of $\mu^*(s)$ to the top of the preference of student s. It is easy to see that when P'_S is the real preference profile, the matching $\mu^*(s)$ will be produced after only one step of the IDAM mechanism.

Another straightforward case is when preferences are common between students.

Remark 7. For any set of students, colleges, capacities, vectors of exam scores, and minimum necessary scores, if all students have the same preference P^* over colleges and follow straightforward strategies, then the IDAM mechanism produces the student-optimal stable outcome after at most k steps, where k is the number of acceptable colleges under P^*.

Finally, we provide an analogous result for the case in which there is a unique grade for each student across all colleges. This is a common situation where, for example, students are ranked based on their grades in a national exam. The result requires a more detailed proof, since as opposed to the remarks above, the mechanics involved when students may have arbitrary preferences are less straightforward.

Proposition 3. *If grades are common across colleges and students follow straightforward strategies, the maximum number of steps it takes for the unbounded IDAM mechanism to produce the student-optimal stable outcome is m.*

The choice of the parameters T^{Max} and π have a great impact on the number of steps needed for a GIDAM mechanism produces an outcome. In fact, if $T^{Max} = 1$ and $\pi(1) = \infty$, the GIDAM mechanism reduces to the usual direct revelation version of the cumulative offer mechanism [Hatfield and Kojima, 2008]. There are many other combinations of these parameters that can be used and that may impact the number of steps, such as the one below.

5.1. **The free-form GIDAM.** The GIDAM family of mechanisms provides a natural way to reduce the number of periods until the Student-Optimal Stable Outcome is produced. It consists of letting students submit rankings over some or all of the options given in their menus at each step. That is equivalent to a GIDAM mechanism where $T^{Max} = \infty$ and for every t, $\pi(t) = \infty$.

In terms of the number of steps, the worst case occurs when students who are given a menu never submit a ranking with more than one option. When some students submit rankings, it makes it possible for the information to be used to make "automatic" choices on these students' behalf. The students who choose to submit rankings have the additional advantage of reducing the likelihood that they will have further interactions with the clearinghouse.

5.2. **The GIDAM+DA alternative.** Another alternative that a policymaker could adopt is to use a hybrid of the iterative mechanisms considered here and the traditional deferred acceptance, which we denote by GIDAM+DA. It consists of running the GIDAM mechanism, with students making only one choice at a time, for a fixed number of steps, and then asking students to submit a ranking over the remaining options. Formally, for a given number of steps $k > 0$, the GIDAM+DA is simply defined as the GIDAM mechanism in which the maximum rank function is such that for all $t \in \{1, \ldots, k\}$, $\pi(t) = 1$, and $\pi(t+1) = \infty$.

One of the main advantages of the GIDAM+DA is that it ends after a number of steps set by the designer: $k + 1$. Moreover, being an unbounded GIDAM, it implements the student-optimal stable outcome in an OPBE of straightforward strategies.

The GIDAM+DA mechanism has some similarities to the German Mechanism: it consists of a dynamic stage followed by a submission of rankings for a deferred acceptance algorithm that terminates the matching. Unlike the German Mechanism, however, it provides robust incentives for students to follow the simple straightforward strategy, in which a stable matching is produced.

5.3. **Simulations.** In this section we present the outcome of several simulations. We compare the number of steps it takes for the unbounded IDAM mechanism with one choice per step to produce the student-optimal stable matching with the length of the rankings that students need to submit so that, for the market in question, truth-telling is an equilibrium.[29] This allows us to have both a quantitative idea of how these two variables relate (time in IDAM vs. length of ranking in DA) and how they perform in each scenario. As we will show, the answer for these questions is not as straightforward as one might think.

[29]The reason we use the constrained DA as a benchmark comparison is that, in effect, almost all centralized clearinghouses use the constrained DA, despite recommendations against including exogenous bounds on the number of choices, as well as experimental evidence against constrained lists Calsamiglia et al. [2010].

The construction of the problems follows a method similar to that applied in Hafalir et al. [2013]. Students' ordinal preferences are derived from utilities that each student has over the colleges. All colleges are deemed acceptable by all students.[30] Student $s \in S$'s utility from being matched to college $c \in C$ is the following:

$$u_s(c) \quad = \alpha\Theta^c + (1-\alpha)\,\Theta_s^c$$

The interpretation of the parameters is as follows. The utility that a student s derives from being matched to a college c is a combination of a value that is shared by all students (Θ^c) and an idiosyncratic value that is unique to a student-college pair (Θ_s^c). The value of Θ^c could therefore be the widespread understanding of the quality of the college and Θ_s^c incorporates, for example, how the college's characteristics fit the student's particular objectives. For each problem, and for each value of $c \in C$ and $(c, s) \in C \times S$, Θ^c and Θ_s^c are independently drawn from the normal distribution with mean zero and variance 1. The value of α, which represents the correlation of preferences between students, is exogenously set in the range $[0, 1]$.

Students' exam grades at each college follow a similar model, and the grade that student s has at college c is:

$$z_c(s) \quad = \beta\Theta^s + (1-\beta)\,\Theta_c^s$$

Here once again, for each problem, the value of Θ^s and Θ_c^s is independently drawn from the normal distribution with mean zero and variance 1. The minimum grade at all colleges is zero (that is, all students are acceptable to all colleges). Moreover, $\beta \in [0, 1]$ is an exogenous parameter that represents the degree of correlation between a student's grades at colleges. Notice that when $\beta = 1$, students have the same exam grade at all colleges. This is the case,

[30] Note that if we allowed for colleges to be deemed unacceptable, since in the IDAM mechanism students submit only one choice from a menu at a time, the number of steps would be weakly fewer than the ones we obtained.

for example, when the criterion used for ranking students is the grade on a single national exam.

It is important to note that the values of u_s and z_c are used in a purely ordinal manner, and therefore the fact that their values might be negative has no relation with the acceptability of the students and/or colleges. In each simulation, we set the values of the parameters (n, m, q, α, β) (where q is the common capacity for all colleges) and generated 20 problems, each representing independent draws for values of the random variables. Every combination of the values of the parameters α and β in steps of 0.1 were used. In other words, every $(\alpha, \beta) \in [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1]^2$ was simulated. In every simulation, the number of colleges (m) was 100 and every college had a capacity of $q = 5$. The number of students was parametrized: $n = \lambda q m$, where λ is the *market balance*, that is, the number of students per seat. Different values of λ allow us to see how the aggregate degree of competition between students for seats affects the results. We used the values $\lambda \in \{0.5, 1.0, 1.5\}$.[31]

Figure 5.1 shows two values for each combination of α and β and λ. In the upper graphs, we show the rounded average number of steps that it took for the unbounded IDAM to produce an outcome. That is, assuming that the time between each step of IDAM is set to be fixed, it represents the total amount of time that it takes for it to end and produce the student-optimal stable outcome. The lower graphs show the rounded average of the maximum of how far in each student's preferences the DA procedure had to go before producing the final outcome. This value, therefore, is the shortest length of the DA ranking in that problem that can guarantee that truth-telling is a Nash equilibrium that yields the student-optimal stable outcome.

Some facts stand out. First, that the value of λ (the market balance) has a substantial qualitative impact on the outcomes. When $\lambda = 0.5$, both variables present a similar behavior: they increase with α and do not change much with β. The fact that the number of steps in IDAM (and similarly the maximum rank in DA) increases with the correlation of preferences

[31]In the Appendix we show the results for the case $\lambda = 5.0$ as a robustness check.

is natural: as preferences become more correlated, many students follow a similar order of applications, with a small number of them being matched to their final allocation at each step.

Some of the most noticeable results are observed when $\lambda = 1$, that is, the number of students equals the number of colleges. Since all colleges and students are mutually acceptable, every student will be matched to a college and, therefore, the mechanisms will just determine the specific match of each student. What we will see is that the performance of IDAM and DA are almost complementary: IDAM performs better when DA is worse, and vice versa. More specifically, when both preferences and grades have low correlation values, IDAM performs especially badly, and DA performs the strongest. The reason for the bad performance of IDAM is that these scenarios are more prone to so-called rejection chains, in which one student applies to a college, which leads to the rejection of another student, who then applies and is tentatively accepted by another college, leading to another rejection, etc. Since grades are not very correlated, the fact that a student was rejected at one college does not correlate with her being rejected at the her next-preferred college, which increases the likelihood of those cycles. When preferences are more correlated, on the other hand, the number of students applying to a college is higher, and that competition makes it less likely that some student will later on displace one who was tentatively matched there. As a result, when α is high, this problem is reduced.

The performance of DA, on the other hand, has a different nature. When the values of α are low, there is less competition overall between the students for each college. As a result, it is possible to satisfy students' preferences to a great extent, matching most of them to their most preferred colleges. When the value of β is high, though, some students have low grades at all colleges, and will therefore end up matched to colleges with seats left empty by other students' choices. As a result, these students will be matched to colleges lower in their preferences, and DA will also perform worse.

Overall, therefore, when it comes to the trade-off between rank length and number of steps when markets are balanced, IDAM and DA excel in complementary scenarios. When the correlation of preferences and grades are low, the execution of IDAM is extended for many additional steps due to a small number of students following rejection chains, whereas in the other scenarios IDAM converges in relatively few steps, especially when grades are correlated.

When $\lambda = 1.5$, however, we see that while DA performs almost just as bad in most configurations, the effect that the rejection cycles have under low correlation of preferences and grades is almost entirely eliminated by the increase in competition between students for the seats.[32] While this increase reduces the likelihood that a student who is rejected from a college is accepted in the next one, it does not change the fact that some students will end up matched to less desirable ones. For the intended application of large-scale national college admissions that use national exams, IDAM therefore presents its highest relative advantage: competition is high, grades are highly correlated, but preferences are less correlated, due to field and geographic preferences.

[32]In the Appendix we show that when the ratio of students per seat is substantially higher, with $\lambda = 5.0$, this difference is even stronger, with IDAM using a smaller number of steps while DA still needs a full ranking to be submitted.

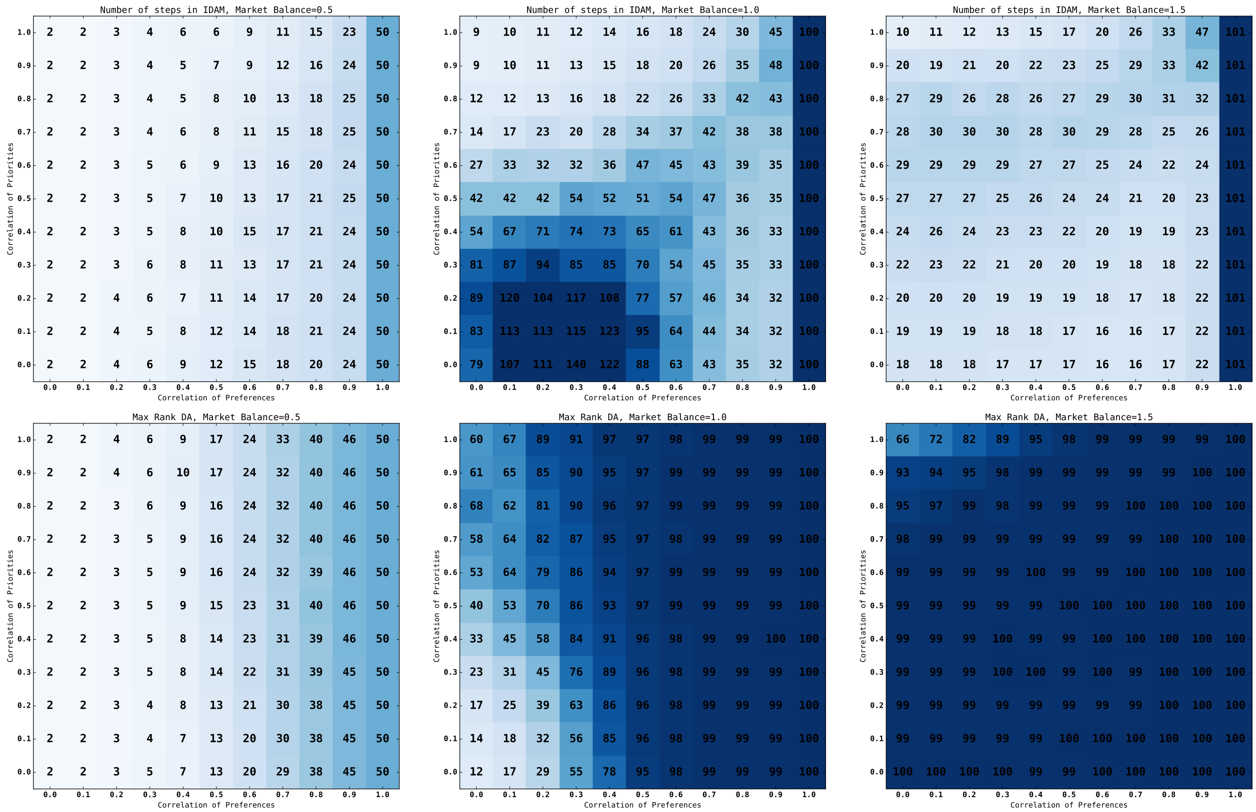

FIGURE 5.1. Number of steps in IDAM vs. maximum rank in DA

Roth and Xing [1997] present simulations of the medical residence matching process that took place in the US for some time. In it, hospitals would sequentially call doctors, following a preference ordering, and doctors accept, reject, or hold on to offers. The process describes an asynchronous version of the college-proposing deferred acceptance. The authors used computer simulations to evaluate the time it would take to end and produce the hospital-optimal stable matching and considered combinations of uncorrelated and fully correlated preferences. There are many reasons why their results cannot be directly compared with our simulations, in particular the fact that the "menu" faced by the hospitals was not updated with the set of doctors who would tentatively accept their offer, as we do in what would be an analogous analysis in our setup. Still, two observations in their exercise have analogous results in our simulations: the longest duration occurred when the proposing side had perfectly correlated preferences and the receiving side had uncorrelated preferences, and the shortest time was observed when the proposing side had uncorrelated preferences and the receiving side had perfectly correlated preferences.

6. Conclusion

In this paper we introduced the GIDAM, a family of iterative mechanisms in which participants are sequentially asked to make choices or submit rankings over menus they are given, and produce stable outcomes in a robust truthful equilibrium.

GIDAM mechanisms are flexible in that they allow for different combinations of number of choices, rankings, and number of steps, but also share with current procedures the dynamic and more transparent way in which the matchings are created.

The GIDAM mechanisms, and the special cases we introduced, improve upon the shortcomings that we identified in current college admissions mechanisms in Germany, Brazil, and Inner Mongolia (China).

While under the German Mechanism accepting offers during the dynamic stage may lead to wastefulness or justified envy (Remark 1), simply choosing the most preferred college at those steps in any GIDAM is part of a robust equilibrium that results in a matching without any justified envy or wastefulness. Moreover, unlike the German Mechanism, under the GIDAM+DA mechanism students submitting truthful rankings after the dynamic steps is also part of that equilibrium (Remark 2 and Theorem 1).

While the Brazilian and Mongolian mechanisms produce unreliable information about the possibility of being accepted at different colleges through cutoff values (Remark 3), under the IDAM mechanism students can safely ignore colleges that are not within reach at any step (Corollary 3). Finally, unlike these two mechanisms (Remark 4), the IDAM mechanism is not subject to manipulations via cutoff (Remark 5).

We believe that there are still many paths to explore on the subject of iterative stable mechanisms. One of them is to use information that the policymaker may have about students' preferences, and optimize the mechanism accordingly. For example, if it is known that a large proportion of the students will have a certain college high in their preferences, the "adaptive" IDAM mechanism could start with a higher initial value for the cutoff at that college, and the stable matching would still be reached, in this case with a high probability.

Another related question involves the design of optimal menus that minimize the amount of information requested from the students, based on the known grade distribution. When grades are common, for example, the IDAM mechanism may obtain information on the preferences that low-grade students have over "top" colleges, but if high-grade students are asked for their preferences earlier, it would not be necessary for this information to be revealed. Ideas similar to these are explored in Ashlagi et al. [2020].

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Appendix A. Appendix

A.1. Formal definitions.

A.1.1. *The generalized iterative deferred acceptance mechanisms (GIDAM).* For any $I \subseteq X$, define $\mathcal{A}_c(I) \equiv \{x \in X \backslash I : x \in f_c(I \cup \{x\})\}$ and $\mathcal{A}_c^s(I) \equiv \mathcal{A}_c(I) \cap I_s$. We denote by $\mathcal{A}_c^s(I)$ the set of **available contracts for** s **at** c **under** I. The interpretation of this is simple: a contract is available for student s at c under I if college c would choose to accept that contract while holding the set of contracts I.

Consider a college matching with contracts market $\langle S, C, T, X, P_S, F_C \rangle$, a maximum number of steps $T^{Max} \in \mathbb{N} \cup \{\infty\}$, a **maximum rank function** $\pi : \mathbb{Z}^+ \rightarrow \mathbb{N} \cup \{\infty\}$, a **public signal set** Θ, and a **public signal function** $\zeta : 2^X \rightarrow \Theta$. The **generalized iterative deferred acceptance mechanism** (**GIDAM**) proceeds as follows:

- **Step** $t = 0$: Let $\mathcal{L}^0 = S$, $S^0 = \emptyset$, and for every $c \in C$, $A^0(c) = \emptyset$. Broadcast the value of $\zeta(\emptyset)$ to the participants. [33]
- **Step** $0 < t \leqslant T^{Max}$:
  - (a) Let $S^t \equiv \{s \in \mathcal{L}^{t-1} | \nexists x \in X_s, c \in C : x \in f_c(A^{t-1}(c))\}$. There are two cases:
    - If $\pi(t) \neq \infty$ or $T^{Max} = \infty$, for every $s \in S$, let the menu of contracts presented to s be $\psi^t(s) \equiv \bigcup_{c \in C} \mathcal{A}_c^s(A^{t-1}(c)) \cup \{\emptyset\}$ if $s \in S^t$ and $\psi^t(s) = \emptyset$ otherwise.
    - If $\pi(t) = \infty$ and $T^{Max} < \infty$, for every $s \in S$, let the menu of contracts presented to s be $\psi^t(s) \equiv \bigcup_{c \in C} \mathcal{A}_c^s(A^{t-1}(c)) \cup \{\emptyset\}$.
  - (b) There are two cases:
    - If $\pi(t) = \infty$ and $T^{Max} < \infty$, request that each student $s \in \mathcal{L}^{t-1}$ submit a ranking of any size of elements in $\psi^t(s)$.
    - If $\pi(t) \neq \infty$ or $T^{Max} = \infty$, request each student $s \in S^t$ to submit a ranking with at most $\pi(t)$ elements in $\psi^t(s)$.

[33] Notation clarification: $\mathcal{L}^t$ is the set of students who are still active at the beginning of step t, and S^t is the set of students who are active and do not have any contract held by a college at the beginning of that step.

– (c) Let, for every student s', $P^t_{s'}$ be the ranking submitted. For every student s'' such that $\psi^t(s'') = \varnothing$, let $P^t_{s''} = P^{t-1}_{s''}$ and, for all $c \in C$, $B^0(c) \equiv A^{t-1}(c)$. Start with $\tau = 0$ and let $\mathcal{L}^t = \mathcal{L}^{t-1}$.

  * Substep $\tau \geqslant 0$: Some student s in $\mathcal{L}^{t-1}$, who does not have a contract held by any college, proposes her most preferred contract with respect to P^t_s, which has not yet been rejected, x. If $x = \varnothing$, remove s from $\mathcal{L}^t$ and from further consideration. Otherwise, college $c(x)$ *holds* x if $x \in \mathcal{A}_c(B^\tau)$, and *rejects* x if $x \notin \mathcal{A}_c(B^\tau)$. Let $B^{\tau+1}(c) = B^\tau(c) \cup \{x\}$ and for all $c' \neq c$, $B^{\tau+1}(c') = B^\tau(c')$.[34]
  * Repeat the process above until no student is able to propose a new contract. Let τ^* be the last step in that process.

– (d) For each college c, let $A^t(c) = B^{\tau^*}(c)$.

– (e) If for every $c \in C$ it is the case that $A^t(c) = A^{t-1}(c)$, stop the procedure.

– (f) Otherwise, broadcast the value of $\zeta\left(\bigcup_{c\in C} A^t(c)\right)$, and proceed to the next step.

- Denote by T^* the last step executed in the procedure. Let $X^* = \bigcup_{c\in C} f_c\left(A^{T^*}(c)\right)$. X^* is the outcome of the GIDAM procedure.

A.1.2. *Extensive-form game formulations and equilibrium concept.* Fix a set of colleges C and their choice functions F_C. The extensive game form G induced by a GIDAM mechanism is a tuple $(S, H, \Phi, P, O, \xi, \pi)$ consisting of:

- A finite set of players $S = \{s_1, \ldots\}$.
- A finite set of actions $A = \{a_1, \ldots\}$.
- A list of preferences over random outcomes $P = (P_{s_1}, \ldots)$.
- A maximum rank function $\pi : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$.

[34]Notice that a contract being in $B^\tau(c)$ does not imply that the contract is held by college c. It simply means that it was offered to c at some step.

- A set of finite histories H, which are a sequence of actions with the property that if $(a_i)_{i=1}^{k} \in H$, then for all $\ell < k$, $(a_i)_{i=1}^{\ell} \in H$. The null history, $h_\varnothing$ is also in H.
- At history h_0, nature draws the value of (X, P) from a joint distribution ξ, and each student s observes the realization of X and of P_s. The distribution ξ is common knowledge.
- Let Z be the set of terminal histories, that is, if $h \in Z$ where $h = (a_i)_{i=1}^{k}$, then there is no $h' \in H$, with $h' = (a'_i)_{i=1}^{\ell}$ where $\ell > k$ and for all $i \leqslant k$, $a_i = a'_i$. Then $(a_i)_{i=1}^{k} \in Z \implies k \mod n = 0$.
- Φ is a player function. $\Phi : H \backslash Z \to S$.[35] There exists an ordering of the players $(s_1, \ldots, s_n)$ such that, for all $h \in H$ such that $|h| \leqslant n$, $\Phi(h) = s_{|h|}$.[36]
  - Let $(a_i)_{i=1}^{k} \in H$, where $k \geqslant 1$. If $(a_i)_{i=1}^{k+n} \in H$, then $\Phi\left((a_i)_{i=1}^{k}\right) = \Phi\left((a_i)_{i=1}^{k+n}\right)$.[37]
- For each student s, $\mathcal{I}_s$ is a partition of $h : \Phi(h) = s$. Define $\zeta\left((a_i)_{i=1}^{k}\right)$ as the list of public signals that result from the sequence of actions in $(a_i)_{i=1}^{k}$ and $\psi_s\left((a_i)_{i=1}^{k}\right)$ as the sequence of menus of contracts presented to student s after that same sequence of actions. Define $H_\ell^t \equiv \left\{(a_i)_{i=1}^{k} \in H : k \mod n = \ell \text{ and } k \div n = t-1\right\}$, and let $h, h' \in H_\ell^t$. The histories $h = (a_i)_{i=1}^{k}$ and $h' = (a'_i)_{i=1}^{k}$ belong to the same member of the partition $\mathcal{I}_{s_\ell}$ if and only if:
  - $|h| \mod n = |h'| \mod n$,
  - $\zeta(h) = \zeta(h')$,
  - $\psi_{s_\ell}(h) = \psi_{s_\ell}(h')$,
  - $a_i = a'_i$ for all i such that $i \mod n = \ell$.[38]
- $A(h)$ are the actions available at $h \in H$. For every $h_i \in H_\ell^t$, the set of actions depend on whether, given the history of actions until step t of the GIDAM mechanism, student

[35] For simplicity, we only allow one player per history. This is without any loss of generality.
[36] That is, the first n actions consist of player s_1 playing first, s_2 second, etc.
[37] Combined with the previous item and the condition on terminal histories, this implies that every player plays every n actions once.
[38] That is, two histories belong to the same set of the partition if the student's preferences are the same, the history of menus faced by the student are the same, the sequence of public signals is the same, and the actions taken by that player were also the same.

$s = \Phi(h_i)$ is offered a non-empty set of contracts, in which case $A(h_i)$ is the set of the ordered list of the contracts in $\psi^t(s)$ with at most $\pi(t)$ elements, or not, in which case we denote $A(h_i) = \{\Diamond\}$, where $\Diamond$ is simply a placeholder for an action when no action is requested from the student. We abuse notation and denote, for any $I_i \in \mathcal{I}_s$, $A(I_i)$ to be $A(h_i)$ for any $h_i \in I_i$ (remember that by definition all histories in I_i have the same set of actions associated with them).

- A strategy for player s is a function $\sigma_s(\cdot)$ that assigns an action in $A(I_i)$ to each information set $I_i \in \mathcal{I}_s$.[39]
- The outcome function O assigns, to each strategy profile $\sigma = (\sigma_{s_1}, \ldots, \sigma_{s_n})$, a random outcome that results from following the histories that result from following those strategies in the GIDAM mechanism, given each realization of X and P.

Since our solution concept will demand that students' strategies are rational at all possible information sets, we will need to consider how students' strategies act at each subgame. We first define a subgame:

Definition 7. A **subgame** of the game G at non-terminal history $h = (a_i)_{i=1}^k$, for $h \in H \backslash Z$ is a game $[G|_h = (S|_h, H|_h, \Phi|_h, P|_h, O)] < G|_h = (S|_h, H|_h, \Phi|_h, P|_h, \pi, O) >$ (we may also abuse notation and let $G|_{I_i} = G|_h$ when $h \in I_i$) where:

- $H|_h = \left\{h' = (a'_i)_{i=k}^l \text{ where } l \geqslant k \text{ and } (a_1, \ldots, a_{k-1}, a'_k, \ldots, a'_l) \in H\right\}$
- $S|_h = \{s \in S : \Phi(h') = s \text{ for some } h' \in H|_h \backslash Z\}$
- $\Phi|_h : H|_h \rightarrow S|_h$ such that for all $h' \in H|_h$, where $h' = (a'_i)_{i=k}^l$, and $\Phi|_h(h') = \Phi(a_1, \ldots, a_{k-1}, a'_k, \ldots, a'_l)$
- For each $s \in S|_h$, $P_s|_h$ satisfies, for all $h', h'' \in H|_h$:

$$h' \, P_s|_h \, h'' \iff (a_1, \ldots, a_{k-1}, a'_k, \ldots, a'_l) \, P_s \, (a_1, \ldots, a_{k-1}, a''_k, \ldots, a''_l)$$

- The weak preference $R_s|_h$ is defined accordingly.

[39]We restrict our analysis to pure strategies.

Finally, let $\sigma|_h = (\sigma_{s_1}|_h, \ldots, \sigma_{s_n}|_h)$ be the strategy profile σ restricted to the subgame $G|_h$. We can define a subgame analogously in terms of an information set instead of a single history. Let t^{∞} be the highest value of t such that $\pi(t) = \infty$. We will consider situations in which students present straightforward behavior. Therefore, we can define a straightforward strategy accordingly:

Definition 8. A strategy σ_s of student $s \in S$ is **straightforward with respect to** P^* if for every t and $h_s^t \in H_s^t$, $\sigma_s(h_s^t|z(s), P^*) = \diamondsuit$ if $A(h_s^t) \neq \{\diamondsuit\}$. Otherwise, $\sigma_s(h_s^t|z(s), P^*)$ consists of the π^* most preferred contracts in $A(h_s^t)$, ordered according to P^*, where $\pi^* \leqslant \pi(t)$ and $\pi^* = |A(h_s^t)|$ when $t = t^{\infty}$.

Let A and B be two random outcomes. We denote by $\gtrdot_s$ the first-order stochastic dominance relation under P_s. That is, $A \gtrdot_s B$ if for all $v \in C \cup \{s\}$, $Pr\{A(s) = v'|v'R_s v\} \geqslant Pr\{B(s) = v'|v'R_s v\}$. A **belief system** ω is a collection of probability measures, one for each information set. Moreover, denote by $O_\omega(\sigma)_G$ the random outcome induced by the strategy profile σ and belief system ω in game G.

Definition 9. A strategy profile σ together with a belief system ω is an **ordinal perfect Bayesian equilibrium (OPBE)** of a game G if:

(i) For every $I_i \in \mathcal{I}_s$ and $s \in S|_{I_i}$, $O_\omega(\sigma_s|_h, \sigma_{-s}|_h)_{G|_{I_i}} \gtrdot_s O_\omega(\sigma'_s|_h, \sigma_{-s}|_h)_{G|_{I_i}}$

(ii) Let $Pr(h|\sigma)$ be the probability that history h is reached, given σ. The belief system satisfies the following property, for any information set I_i that is reached with positive probability, and $h \in I_i$: $\omega(h) = \frac{Pr(h|\sigma)}{\sum_{h' \in I_i} Pr(h'|\sigma)}$.

A.2. **Proofs.**

Lemma 1.

Proof. First, note that if $\psi^{t'}(s) = \varnothing$, then the statement is true. Suppose, for the sake of contradiction, that $\psi^{t'}(s) \neq \varnothing$ and the statement is false. Then, there is a student $s \in S$, $0 \leqslant t \leqslant t' \leqslant T^*$ and a contract $x^* \in X$ such that $x^* \in \psi^{t'}(s)$ but $x^* \notin \psi^t(s)$. Since for any

$I' \subseteq X$, $\mathcal{A}_c(I) \subseteq I_c$, we can separate the violation of the lemma into the contracts available for a student from a single college. There is, therefore, a college c such that:

$$x^* \in \mathcal{A}_c^s\left(A^{t'}(c)\right) \text{ but } x^* \notin \mathcal{A}_c^s\left(A^t(c)\right)$$

By construction, for every $c \in C$, $A^t(c) \subseteq A^{t'}(c)$. Therefore:

$$x^* \in \mathcal{A}_c^s\left(A^{t'}(c) \cup \left[A^{t'}(c) \backslash A^t(c)\right]\right) \text{ but } x^* \notin \mathcal{A}_c^s\left(A^t(c)\right)$$

And given the definitions of $\mathcal{A}_c^s$ and $\mathcal{A}_c$:

$$x^* \in f_c\left(A^t(c) \cup \left[A^{t'}(c) \backslash A^t(c)\right] \cup \{x^*\}\right) \text{ but } x^* \notin f_c\left(A^t(c) \cup \{x^*\}\right)$$

We need to consider three cases: (i) $x^* \notin A^{t'}(c) \cup A^t(c)$, (ii) $x^* \in A^t(c)$, and (iii) $x^* \in A^{t'}(c) \backslash A^t(c)$. Cases (i) and (ii): In both cases, $x^* \notin \left[A^{t'}(c) \backslash A^t(c)\right]$. Since f_c satisfies IRC, $f_c\left(A^t(c) \cup \left[A^{t'}(c) \backslash A^t(c)\right] \cup \{x^*\}\right) = f_c\left(A^t(c) \cup \{x^*\}\right)$. But this contradicts $x^* \notin f_c\left(A^t(c) \cup \{x^*\}\right)$. Case (iii): Here we will use the following claim, which can easily be derived from the definition of unilateral substitutes:

If contracts are unilateral substitutes for college c under f_c, there does not exist contract $z \in X_s$ and sets of contracts $Y \subseteq X \backslash X_s$ and $I \subseteq X \backslash X_s$ such that $z \notin f_c(Y \cup \{z\})$ and $z \in f_c(Y \cup I \cup \{z\})$.

Denote by $I^* = \left[A^{t'}(c) \backslash A^t(c)\right] \backslash \{x^*\}$. Then:

$$x^* \in f_c\left(A^t(c) \cup I^* \cup \{x^*\}\right) \text{ but } x^* \notin f_c\left(A^t(c) \cup \{x^*\}\right)$$

By IRC and the fact that f_c chooses only one contract per student:

$$x^* \in f_c\left(\left[A^t(c) \backslash X_s\right] \cup \left[I^* \backslash X_s\right] \cup \{x^*\}\right) \text{ but } x^* \notin f_c\left(A^t(c) \backslash X_s \cup \{x^*\}\right)$$

Following the claim above, this contradicts the assumption that f_c satisfies unilateral substitutes, finishing the proof. □

Proposition 1.

Proof. First, note that given the description of an unbounded GIDAM mechanism and Lemma 1, every time a student is asked to submit a ranking, the set of contracts available under a GIDAM mechanism is weakly smaller. Moreover, for any t, t' and $s \in S$ such that $0 \leqslant t < t' \leqslant T^*$, $\psi^t(s) \neq \varnothing$ and $\psi^{t'}(s) \neq \varnothing$, it must be that the set of contracts in $\psi^{t'}(s)$ is a strict subset of $\psi^t(s)$, since at least the highest-ranked contract submitted by student s in step t must have been rejected by step t'. Therefore, in every step the set $\psi^t(s)$ is strictly smaller for at least one student. Since X is finite, GIDAM will end and will produce an outcome after a finite number of steps.

Next, notice that regardless of which straightforward strategy students use, in all of them students will offer contracts following the order of their preferences, perhaps only skipping those that would not be held by the college associated with the contract, and that the outcome will be produced when every student either chooses $\varnothing$, has a contract held by a college, or reaches the end of the last ranking submitted. According to Hirata and Kasuya [2014], if the choice functions (cumulative offer) process will produce the student-optimal stable matching regardless of the order in which doctors are called to offer contracts, as long as the order in which each student offers her contracts follow their preferences over them. Different straightforward strategies may imply different orders in which students offer contracts, but this does not change the fact that students follow their own preference until the end.[40] Therefore, for any profile of straightforward strategies, the outcome of the GIDAM mechanism will always be the student-optimal stable matching. □

Proposition 2.

Proof. For this proof we consider an exam-based college matching market and an IDAM mechanism with $\pi(t) = 1$ for all t. Consider the set of students $S = \{s_1, s_2, s_3\}$ and of

[40]Technically speaking, under the cumulative order process students will always offer contracts following their preferences, even those that would not be accepted by the college in the contract. Since choice functions satisfy IRC, however, this is equivalent to a process that simply skips those contracts that would not be accepted (and are, therefore, not part of the menus offered to the students under the GIDAM mechanism).

colleges $C = \{c_1, c_2, c_3\}$, each with capacity $q_i = 1$. Student s_1—who will be the player to whom we will show having no dominant strategy—has preferences $c_1 P_{s_1} c_2 P_{s_1} c_3$, and students' exam grades at those colleges are as follows:

	c_1	c_2	c_3
s_1	100	100	100
s_2	200	200	200
s_3	300	300	300

Suppose now that, conditional on the realized preferences and grades of student s_1, student s_3 follows a straightforward strategy with respect to the preference $c_3 P^3 c_2 P^3 c_1$. Notice that we are not stating that those are the preferences of student s_3, we are simply assuming that she will follow the *straightforward strategy with respect to* P^3. Next, we consider two strategies for student s_2 and show that no strategy for s_1 is a common best response for these two possibilities.

Scenario 1

Suppose that student s_2's strategy is as follows: in $t = 1$, choose c_3. If at some later point s_2 is asked again to make a choice, she will choose the college with the highest cutoff value at that step among the options available. In the event of a tie, she will choose the college with the lowest index number (for example, the index number of c_2 is 2). We will show that, given s_2 and s_3's strategies, the best response for s_1 involves first choosing c_2. The sequence of steps will be as follows:

Step 1: Student s_1 applies to c_2. Students s_2 and s_3 apply to c_3. Student s_2 is rejected. Cutoffs $\left(\zeta^1_{c_1}, \zeta^1_{c_2}, \zeta^1_{c_3}\right)$ are $(0, 100, 300)$.

Step 2: Since $\zeta^1_{c_2}$ is the highest cutoff among the colleges offered to s_2, student s_2 applies to c_2. Student s_1 is rejected. Cutoffs $\left(\zeta^2_{c_1}, \zeta^2_{c_2}, \zeta^2_{c_3}\right)$ are $(0, 200, 300)$.

Step 3: Student s_1 is left with two options: choose c_1 or s. If she chooses s, she will remain unmatched. If she applies to c_1, she will be accepted. Final cutoffs $\left(\zeta^3_{c_1}, \zeta^3_{c_2}, \zeta^3_{c_3}\right)$ would then be $(100, 200, 300)$ and the outcome would be the matching μ' as follows:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 \\ \\ s_1 & s_2 & s_3 \end{pmatrix}$$

Student s_1 can therefore be matched to her most preferred college by first choosing c_2. Next, we show that by choosing first c_1 or c_3, s_1 will always be matched to a strictly inferior college. First, let her initially choose c_1:

Step 1: Student s_1 applies to c_1. Students s_2 and s_3 apply to c_3. Student s_2 is rejected. Cutoffs $\left(\zeta^1_{c_1}, \zeta^1_{c_2}, \zeta^1_{c_3}\right)$ are $(100, 0, 300)$.

Step 2: Since $\zeta^1_{c_1}$ is the highest cutoff among the colleges offered to s_2, student s_2 applies to c_1. Student s_1 is rejected. Cutoffs $\left(\zeta^2_{c_1}, \zeta^2_{c_2}, \zeta^2_{c_3}\right)$ are $(200, 0, 300)$.

Step 3: Student s_1 is left with two options: choose c_2 or s. If she chooses s she will remain unmatched. If she applies to c_2, she will be accepted. Final cutoffs $\left(\zeta^3_{c_1}, \zeta^3_{c_2}, \zeta^3_{c_3}\right)$ would then be $(200, 100, 300)$ and the outcome would be the matching μ' as follows:

$$\mu' = \begin{pmatrix} c_1 & c_2 & c_3 \\ \\ s_2 & s_1 & s_3 \end{pmatrix}$$

If s_1 chooses c_3 first instead, the following will happen:

Step 1: Students s_1, s_2, and s_3 apply to c_3. Students s_1 and s_2 are rejected. Cutoffs $\left(\zeta^1_{c_1}, \zeta^1_{c_2}, \zeta^1_{c_3}\right)$ are $(0, 0, 300)$.

Step 2: Following her strategy and the fact that college c_1's index is lower than c_2, student s_2 applies to c_1. Student s_1 has three options: also choose c_1 and therefore be rejected and left to choose between c_2 and s in step $t = 2$, choose c_2, or choose s. In all cases she will either end up remaining unmatched or matched to c_2.

Scenario 2

Now suppose that student s_2 follows a similar strategy to scenario 1, but where instead of applying to c_3 and then to the college with the highest cutoff value, she applies to the college with the *lowest* cutoff value, once again breaking ties based on the index of the college.[41] Following an exercise similar to the one above, it is easy to see that student s_1's strategies that involve choosing c_2 or c_3 first will lead to her either being unmatched or matched to c_2, while choosing c_1 will match her to c_1, her most preferred college.

Since every best response strategy under scenario 1 is dominated by different strategies in scenario 2, we have shown that a student may not have a weakly dominant strategy for the game induced by the IDAM mechanism, and as a consequence also the GIDAM mechanisms. □

Theorem 1.

Proof. We use the extensive game notation introduced in Appendix A.1. Consider some history $h \in H$. Given other players' strategies σ_{-s}, the history that results from the strategy profile (σ_s, σ_{-s}) consists of a series of steps in which each student has either only the action $\diamondsuit$ or some menu of options $\psi^t(s)$ and a maximum rank $\pi(t)$, as described in the definition of the GIDAM. Therefore, given our strategy profile and student s, we can write down a list of pairs, with menus given to student s and her submitted ranking.

Suppose that the sequence of menus offered and actions chosen for a student s up to history h are as follows:

$$\left(\left(\psi^1, a^1\right), \left(\psi^2, a^2\right), \ldots, \left(\psi^t, a^t\right)\right)$$

For simplicity and without any loss of generality, assume that the sequence above has removed from it the list of pairs $(\varnothing, \diamondsuit)$. We show below that menus given to students never include contracts present in any previously submitted ranking.

[41]Although the strategies used in this proof for student s_2 may seem very arbitrary, they can be rationalized by two simple stories. Student s_2's strategy in scenario 1 is consistent with a student who knows that her top choice is c_3 but has some uncertainty about whether c_1 or c_2 is her second choice, and sees the cutoff grade as an indication of how competitive acceptance is at those colleges and therefore sees the perceived quality of those. The strategy in scenario 2 could be rationalized by a student who once again knows that her top choice is c_3 but who would otherwise prefer to go to a college with low-achieving peers, and uses the low cutoff as an indication of that fact.

Claim. If contract x is in a^t, then for every t' such that $t' > t$, $x \notin \psi^{t'}(s)$.

Proof. Let $c = c(x)$. If $\psi^{t'}(s) = \{\diamondsuit\}$, the claim obviously holds. Therefore, we consider the case in which both $\psi^t(s)$ and $\psi^{t'}(s)$ have a positive number of contracts available. Since x is in a^t, $x \in \psi^t(s)$. Also, since $x \in a^t$ and the fact that $\psi^{t'}(s) \neq \{\diamondsuit\}$, it must be the case that all the contracts in a^t are in $A^{t'-1}(c)$ (otherwise the GIDAM mechanism would still use the ranking a^t in step t'). Also, by the definition of the GIDAM, $\nexists y \in X_s, c \in C : y \in f_c\left(A^{t'-1}(c)\right)$, and in particular $x \notin f_c\left(A^{t'-1}(c)\right)$. Therefore, $x \notin \psi^{t'}(s)$. □

Therefore, there is no repetition of contracts in a^i, $i = 1, \ldots, t$. We will abuse notation and use a^i to represent the student's choice both as a ranking and as a set of contracts. Denote $\psi^i_- \equiv \psi^i \backslash \bigcup_{j=i}^t a^j$ and $X_s^+ \equiv X_s \backslash \{\varnothing\}$. We will show that this sequence could have been generated by a student's straightforward strategy with a preference relation in the following class of preferences:[42]

$$X_s^+ \backslash \psi^1 \; R_s^* \; a^1 \; P_s^* \; \psi^1_- \backslash \psi^2_- \; R_s^* \; a^2 \; P_s^* \; \psi^2_- \backslash \psi^3_- \; R_s^* \cdots R_s^* \; a^t \; P_s^* \; \psi^t_-$$

The notation above includes a class of strict preferences because some of its elements ($X_s^+ \backslash \psi^1$, $\psi^1_- \backslash \psi^2_-$, etc.) consist of (possibly empty) sets of contracts. Any strict preference derived from some ordering over the elements of each of those sets belongs to the class of preferences that we are referring to. We will use P_s^* to refer to some arbitrary element of those preferences. The claim below implies that each preference in that class is complete over the set of contracts and that no contract appears more than once.

Claim. $\psi^t_- \subseteq \psi^{t-1}_- \subseteq \cdots \subseteq \psi^1_- \subseteq X_s$, and $a^i \cap \psi^j_- = \varnothing$ for all i, j.

Proof. First, note that $\psi^1_- = \psi^1 \backslash \bigcup_{j=1}^t a^j$. Since by definition ψ^1 is nonempty, $a^1 \subseteq \psi^1$, and since by definition $\psi^1 \subseteq X_s$, it follows that $\psi^1_- \subsetneq X_s$. By the definition of ψ^t and Lemma 1, $\psi^k \subsetneq \psi^{k-1}$. Therefore:

[42] Note that this class of preferences does not necessarily include all the preferences that are compatible with the choices made.

$$\psi^{k-1}\backslash \bigcup_{j=k-1}^{t} a^j = \left(\left(\psi^{k-1}\backslash\psi^k\right) \cup \psi^k\right) \backslash \left(a^{k-1} \cup \bigcup_{j=k}^{t} a^j\right)$$

Consider now any $k > 1$. By definition, all contracts in a^{k-1} are in ψ^{k-1}, and by the first claim in this proof, no contract in a^{k-1} is in ψ^k. Therefore:

$$\psi_-^{k-1} = \psi^{k-1}\backslash \bigcup_{j=k-1}^{t} a^j = \left(\left(\left(\psi^{k-1}\backslash\psi^k\right) \backslash a^{k-1}\right) \cup \psi^k\right) \backslash \bigcup_{j=k}^{t} a^j = \psi_-^k \cup \left(\left(\psi^{k-1}\backslash\psi^k\right) \backslash a^{k-1}\right)$$

That is, $\psi_-^{k-1} = \psi_-^k \cup \left(\left(\psi^{k-1}\backslash\psi^k\right) \backslash a^{k-1}\right)$, which implies that $\psi_-^k \subseteq \psi_-^{k-1}$. Finally, for every $j \geqslant i$, it follows from the definition of ψ_-^i that $a^j \cap \psi_-^i = \varnothing$. Suppose instead that there is a $i > j$ such that $a^j \cap \psi_-^i = I$, for some nonempty set of contracts I. In that case, the definition of ψ_-^i implies that $I \subseteq \psi^i$. But in that case, we have that the contracts in I were submitted in a ranking by the student in step j and were available in the menu in step $j > i$, which contradicts the first claim in the proof.

Next, take some of the menus that were offered, ψ^i. We now show that for all $a \in \psi^i$ where $a \notin a^i$, $a^i P_s^* a$. For this, it suffices to show that:

$$a \in \bigcup_{j=i+1}^{t} a_j \cup \bigcup_{j=i}^{t-1} \psi_-^j\backslash\psi_-^{j+1} \cup \psi_-^t$$

That is, we will show that a must be at some element to the right of those in a^i in the definition of P_s^*. Since $a \notin a^i$, this is equivalent to:

$$a \in \bigcup_{j=i}^{t} a_j \cup \bigcup_{j=i}^{t-1} \psi_-^j\backslash\psi_-^{j+1} \cup \psi_-^t$$

Since we defined $\psi_-^i \equiv \psi^i\backslash \bigcup_{j=i}^{t} a^j$, we can rewrite the condition as:

$$a \in \underbrace{\psi^i\backslash\psi_-^i}_{(i)} \cup \underbrace{\bigcup_{j=i}^{t-1} \psi_-^j\backslash\psi_-^{j+1}}_{(ii)} \cup \underbrace{\psi_-^t}_{(iii)}$$

Suppose not. Then a cannot be in (i), (ii), or (iii). By (i), it must be that $a \notin \psi^i \backslash \psi^i_-$. Since $a \in \psi^i$, that implies $a \in \psi^i_-$. By (ii), since $a \notin \psi^i_- \backslash \psi^{i+1}_-$, it must then be that $a \in \psi^{i+1}_-$. This reasoning can be repeated until finding that it must be that $a \in \psi^t_-$. But this is (iii), which leads to a contradiction. □

The sequence $((\psi^1, a^1), (\psi^2, a^2), \ldots, (\psi^t, a^t))$ is consistent with student s having a preference over contracts P^*_s and following a straightforward strategy that in each step $k \leqslant t$ submits a ranking with the top $|a^k|$ contracts among those available, with respect to her preference.

This implies that, since all other students follow straightforward strategies, every deviating strategy for student s is outcome-equivalent to following a straightforward strategy for some preference over contracts that is not necessarily that student's real preference P_s. Proposition 1, therefore, shows that the outcome produced will be the student-optimal stable matching with respect to the preference profile (P^*_s, P_{-s}). Theorem 7 in Hatfield and Kojima [2010] shows that since colleges' choice functions satisfy unilateral substitutes and the law of aggregate demand, submitting a true ranking is always a best-response when using a direct mechanism. In light of the above result, when other students follow straightforward strategies, any deviating strategy for s is outcome-equivalent to a deviating strategy in the direct mechanism, and as a result is not profitable. Therefore, for every belief system, given that other players' strategies σ_{-s} are straightforward, following any straightforward strategy is a best-response for student s.

We have two more steps to follow. The first one shows that, for any system of belief ω, deviating strategies are stochastically dominated by straightforward ones under this equilibrium. Since we focus on pure strategies, the only source of uncertainty is the draw of P and X that takes place in history h_0. The fact that a truthful ranking is always a best-response in the direct mechanism that yields the student-optimal stable matching implies that, regardless of other students' preferences (and the set of contracts X), the outcome that a student obtains by using the true preference is always weakly better than any other strategy. In particular, this implies that following any straightforward strategy will yield her most preferred contract whenever there exists a strategy that yields that while the realization of other students' preferences makes it possible. Also, due to the strategy-proofness in the direct mechanism, straightforward strategies will always match a student with the second most preferred contract whenever X and other students' preferences are such that the first is not possible and the second is for some strategies. This can be done for every contract in the student's preference, and proves that any straightforward strategy stochastically dominates any deviating strategy.

Finally, we consider the fact that the definition of OPBE implies that deviating strategies are stochastically dominated by straightforward strategies *starting from any information set*. Obviously, any deviation that is outcome-relevant effectively starts from an information set in which a student receives a nonempty menu. Let t be the first step in which student s is given a nonempty menu while using a deviating strategy.

Consider now the extensive-form game that is induced by the GIDAM mechanism in which all contracts that were rejected at some step before t *are removed from* X. Let us call this $\bar{X}$. In the first step of that game, all students apply to their most preferred contract in $\bar{X}$. By Hirata and Kasuya [2014], this is equivalent to, first, all students who had a contract being held by a college in time t applying first, and then letting the students who are offered a menu at time t to apply afterwards. By definition of $\bar{X}$ and the fact that students preferences are used from the top to the bottom, the contracts being held by colleges in period t are the most preferred contracts in $\bar{X}$ by those students. By IRC, all of the contracts offered by the students who had contracts held in step t will have those contracts accepted. Therefore, when the students who were given menus in step t make their applications, the set of contracts being held by colleges is the same as those being held in the original game, at step t.

Therefore, we have that starting from any history in which a student is given a menu to choose from, the continuation game is equivalent to a game induced by the GIDAM mechanism, where the set of contracts is $\bar{X}$. Given that other players follow straightforward strategies starting at that history, this implies that any deviating strategy is stochastically dominated by following straightforward strategies. □

Proposition 3.

Proof. We will show that at each step of the IDAM mechanism, at least one college will have its final matching determined. In order to make the result stronger, we will assume that all colleges are acceptable to all students.[43] Let z_C denote the common exam score used by all colleges.

Claim: For every period t and $t+1$ in which there are students who were rejected, the highest grade among those students who were rejected at t is higher than the highest grade among those students who were rejected at $t+1$.

Proof: Suppose not. Then there is a student s who is rejected at period $t+1$ such that $z_C(s) > z_C(s')$ for any student who was rejected at period t. Clearly, s could not have been rejected at period t. Let c be the college s is tentatively matched to by the end of period t. In order for s to be rejected from c, there must be at least one student with a grade higher than $z_C(s)$ who applied to c in period $t+1$. But that contradicts the assumption that all students rejected at t have grades below $z_C(s)$.□

Now, consider the period $t=1$. There are two cases, one where at the end of the first step the cutoffs for all colleges are either zero or the minimum necessary score.[44] In this case, no student was rejected, and therefore the IDAM mechanism ends, matching all students to their choices. Otherwise, let ζ^{1*} be the highest cutoff value of a college at the end of the period $t=1$, and denote that college by c^{1*}. Clearly, under the definition of cutoff values, no student who had their choice in $t=1$ rejected has an exam grade higher than ζ^{1*}. Moreover, under the claim above, in any period that follows, no student who is rejected has a grade higher than ζ^{1*}. Therefore, no student will have c^{1*} in their menus after period $t=1$. Therefore, c^{1*}, and potentially other colleges, are matched to their final matches.

[43]If some students deem some colleges unacceptable, this obviously cannot increase the number of steps the IDAM mechanism takes before producing the final outcome.

[44]This would typically only occur when the number of seats at all colleges exceeded the number of students.

For every period $t > 1$, the same reasoning follows. If no cutoff has changed, then the final allocation was reached. Otherwise, among the colleges that did not reach their final allocation, let ζ^{t*} be the highest cutoff value, associated with college c^{t*}. Following the same argument above, no student who is rejected after period t has a grade higher than ζ^{t*}, and therefore c^{t*} will not be in a student's menu after t. Therefore, c^{t*}, and potentially other colleges, are matched to their final matches.

Since at every step at least one college is matched to their final matches (and does not appear in any menu), and since there are m colleges, the maximum number of steps it takes for the unbounded IDAM to produce the student-optimal stable outcome is m. □

A.3. **Additional simulation data.**

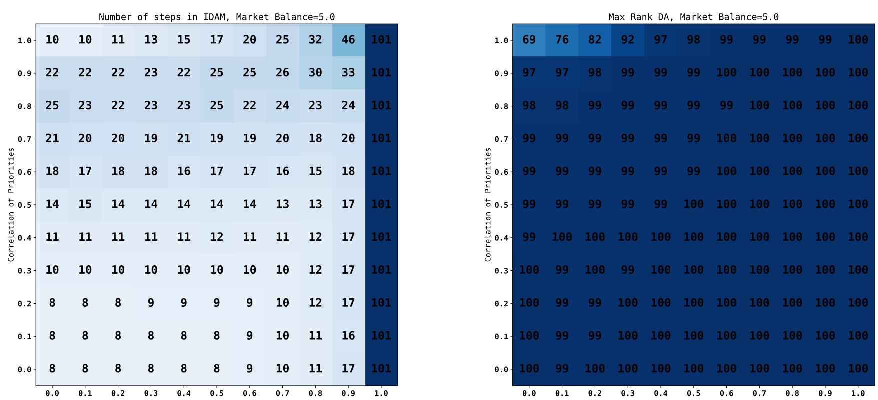

FIGURE A.1. Number of steps in IDAM vs. maximum rank in DA

A.3.1. *Number of steps in IDAM vs. maximum rank in DA when market balance is 5.0.*