The Iterative Deferred Acceptance Mechanism*

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October 21, 2016

PRELIMINARY AND INCOMPLETE

Abstract

We introduce a new mechanism for matching students to schools or universities, denoted Iterative Deferred Acceptance Mechanism (IDAM), inspired by a procedure currently being used to match millions of students to public universities in Brazil. Differently from most options available in the literature, IDAM is not a direct mechanism. Instead of requesting from each student a full preference over all colleges, she is instead repeatedly asked to choose one college among those which would accept her given the current set of students choosing that college. Although the induced sequential game has no dominant strategy, when students simply choose the most preferred college in each period (denoted the *straightforward strategy*), the matching that is produced is the Student Optimal Stable Matching. Moreover, under imperfect information, students following the straightforward strategy is an Ordinal Perfect Bayesian Equilibrium. We also provide evidence that, due to shortcomings that are absent in the modified version that we propose, the currently used mechanism in Brazil fails to assist the students with reliable information about the universities that they are able to attend, and are subject to manipulation via cutoffs, a new type of strategic behavior that is introduced by this family of iterative mechanisms.

classification: C78, C92, D63, D78, D82
Keywords: Market Design, Matching, Iterative Mechanisms,College Admissions.

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*We would like to thank Samson Alva, Nick Arnosti, Orhan Aygün, Nina Bonge, Li Chen, Sombuddho Ghosh, Maria Godinho, Robert Hammond, PJ Healy, C.-Philipp Heller, Bettina Klaus, Dorothea Kübler, Alexander Nesterov, Alvin Roth, Bertan Turhan, Utku Ünver, Yosuke Yasuda, Alexander Westkamp and participants at the Berlin Behavioral Economics Workshop 2016, Matching in Practice Workshop 2016 and seminars at the National University of Singapore and ITAM.

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1 Introduction

The use of central clearinghouses for matching prospective students to colleges or schools[1], as well as theoretical and empirical studies of those procedures that are used have been steadily increasing over the last decades. The vast majority of them consists of procedures in which students submit rank ordered lists of colleges, colleges submit rankings over students (many times based on the students' grades in a national exam) and a matching of students to colleges is produced. If we consider the students to be the relevant strategic agents who interact with the procedure and take the colleges' ranking over students as given (based solely in an exam score, for example), the mechanisms used are in general, therefore, direct mechanisms, in which the information provided by the students is their preferences over colleges.

One of the common objectives of policy makers is for the matching generated to be fair, that is, matchings in which the reason a student may not be matched to a more preferred college is that all students who are matched to that college have higher priority than her. Balinski and Sönmez [1999] showed that the Gale-Shapley student proposing deferred acceptance procedure (DA) is characterized as the "best" fair mechanism, in that it is strategy-proof and Pareto dominates any other fair mechanism (that is, it is constrained efficient). In fact, variations of the DA mechanism are used in many real-life student matching programs around the world. College and secondary school admissions in Hungary [Biró, 2012], high school admissions in Chicago [Pathak and Sönmez, 2013] and New York City [Abdulkadiroğlu et al., 2009] as well as elementary school in Boston [Abdulkadiroglu et al., 2006] are examples of the use of the DA mechanism.

Other mechanisms, such as the college proposing DA, top trading cycles, the so-called "Boston mechanism" and the "Shanghai mechanism" are used to match millions of students to schools and colleges around the world [Chen and Kesten, 2015, Abdulkadiroğlu and Sönmez, 2003, Balinski and Sönmez, 1999].

In this paper, we analyze a mechanisms currently being used to match students to public universities in Brazil, denoted SISU. The SISU mechanism differs from most other ones analyzed in the literature in that it doesn't require students to submit rank ordered lists over colleges, but instead provides information about tentative requirements for acceptance at each university and asks students to choose one college among them, producing an allocation after a fixed number of periods. We show that the SISU mechanism has some undesirable theoretical properties: it fails to give reliable information about where students could be accepted, and is subject to a new type of manipulation, denoted manipulation via cutoffs. We show, based on data obtained from the selection process that took place in 2016, that the first problem is empirically relevant, and provide anecdotal evidence that manipulation via cutoffs takes place in real life.

We then propose a new mechanism for matching students to colleges, denoted Iterative Deferred Acceptance Mechanism (IDAM), based on a few simple modifications of the SISU mechanism. In each step of the IDAM mechanism, the period-specific acceptance requirement, in the form of a cut-off value for each university, is made public. Students who are not

[1] From this point on, we will refer to the institutions as colleges. Unless explicitly stated, all arguments also hold for schools.

tentatively assigned to some college are given the option to choose from a menu of universities where the acceptance requirement in that period is such that the student would be accepted. At the end of each period, students' choices and tentative allocations are combined for each university and as a result some students may be rejected, being therefore given the choice to make another choice in the next period. An allocation is produced after a period in which no student is rejected. If students follow the simple strategy of choosing the most preferred college among those available at each step of the IDAM mechanism (denoted the *straightforward strategy*), the matching produced as an outcome is the Student Optimal Stable Matching, that is, the matching that is the most preferred by all students among all stable matchings.

While, unlike the standard Gale-Shapley Student-Proposing Deferred Acceptance (DA) mechanism, the IDAM doesn't have a dominant strategy, we show that stable outcomes are equilibrium outcomes under both imperfect and perfect information under a robust equilibrium concept. More specifically, under imperfect information about other players' preferences and exam grades, following a straightforward strategy first-order stochastically dominates any other strategy at every subgame, when other players follow the straightforward strategy. Although under some extreme scenarios the number of steps that the mechanism takes until producing the allocation may be relatively high, we also show that if the number of steps is limited and students still follow the same strategy, the number of students involved in blocking pair falls very quickly at each step.

Proofs absent from the main text can be found in the appendix.

2 Model

An **college matching market** is a tuple $\langle S, C, q, P_S, P_C \rangle$:

1. A finite set of **students** $S = \{s_1, \ldots, s_n\}$,
2. A finite set of **colleges** $C = \{c_1, \ldots, c_m\}$,
3. A **capacity vector** $q = (q_{c_1}, \ldots, q_{c_m})$,
4. A list of strict **student preferences** $P_S = (P_{s_1}, \ldots, P_{s_n})$ over $C \cup \{s\}$[2],
5. A list of strict **college preferences** over sets of students $P_C = (P_{c_1}, \ldots, P_{c_m})$[3]

An **exam-based college matching market** consists of a college matching market where:

1. Colleges have **vectors of exam scores** $z = (z(s_1), \ldots, z(s_n))$, where for each $s \in S$, $z(s) = (z_{c_1}(s), \ldots, z_{c_m}(s))$, are the exam scores that student s obtained, respectivelly, at college $c_1 \ldots, c_m$. We assume that for every $s, s' \in S$ and $c \in C$, $z_c(s) = z_c(s') \implies s = s'$,

[2]Here s represents a student remaining unmatched to any college.

[3]Whenever adequate, we abuse notation in the notation of preferences oves singleton sets as sP_cs' instead of $\{s\} P_c \{s'\}$.

2. Colleges have **minimum necessary scores** $\underline{Z} = \left(\underline{z}_{c_1}, \ldots, \underline{z}_{c_m}\right)$.

3. Colleges' preferences over sets of students are **responsive to exam scores**, that is, for all $c \in C$ and $I \subseteq S$ such that $|I| < q_c$:

   (a) For all $s, s' \in S \backslash I$, $I \cup \{s\} P_c I \cup \{s'\} \iff z_c(s) > z_c(s')$,

   (b) For all $s \in S$, $I \cup \{s\} P_c I \iff z_c(s) \geqslant \underline{z}_c$.

We can also represent an exam-based college matching market by the tuple $\langle S, C, q, P_S, P_C, \underline{Z}, z \rangle$. The preference relation P_s for student s is over the set of colleges and the option of remaining unassigned, that is, $C \cup \{s\}$. Given a strict preference relation P_s, we can also derive the corresponding weak preference relation R_s, where $cR_sc' \iff cP_sc'$ or $c = c'$. We say that student s is **acceptable** for college c if $\{s\} P_c \varnothing$. We say that college c is acceptable for student $s$if cP_ss.

A **matching** μ is a function from $C \cup S$ to subsets of $C \cup S$ such that:

- $\mu(s) \in C \cup \{s\}$ and $|\mu(s)| = 1$ for every student s[4],
- $\mu(c) \subseteq S$ and $|\mu(c)| \leqslant q_c$ for every college c,
- $\mu(s) = c$ if and only if $s \in \mu(c)$.

A matching is **individually rational** if for every student s, $\mu(s) P_s s$ and for every college $\mu(c) P_c \varnothing$. A matching μ is **blocked** by a student s and college c if student s is acceptable to college c, $cP_s\mu(s)$ and either $|\mu(c)| < q_c$ or there is a student $s' \in \mu(c)$ where $(\mu(c) \cup \{s\}) \setminus \{s'\} P_c \mu(c)$. A matching $\mu$is **stable** if it is individually rational and is not blocked. In this model, as opposed to some of the literature in college admissions, colleges are not considered strategic. Since the actual real life examples in which this family of mechanisms was observed were for college admissions where the rules determining admission criteria were decided by governments, that assumption fits the applications in mind, although it makes this problem closes to the assumptions in the school choice literature.

3 The SISU mechanism

Until 2010, college admissions in Brazil were essentially decentralized, with no central mechanism matching students to the programs in universities. Starting in 2010, a new method for matching students to university programs[5] was introduced by the ministry of education, denoted SISU. The SISU system represented a significant change in the way by which universities admitted students. First, it unified the acceptance criteria at the universities for the seats made available through the system: instead of a different exam for each university,

[4] We abuse notation and consider $\mu(s)$ as an element of $C \cup \{s\}$, instead of a set with an element of $C \cup \{s\}$.

[5] Differently from countries like the US, in Brazil a student is accepted to a specific program in a University (for example, Economics at the University of Brasilia).

a unified national exam was used.[6] Second, students were free to apply to any program in any university in the country (among those available in the SISU) without any extra cost, whereas before in some cases the student would have to travel to the location of the university just to be able to apply. Third, and perhaps more importantly, the centralized system could allow a student to obtain information about which university programs would accept him.

During the period of 2010 to 2016, the precise rules which define the SISU mechanism were changed multiple times. The version that we will consider for analysis is the one used for the year of 2010, due to its simplicity, so whenever we refer to the *SISU mechanism* we are referring to this version of it. Although later versions have different modifications, as far as we know all the problems identified in this section are also present in the later versions.

The mechanism runs for 4 days. During the entire day t, for $t = 1, \dots 4$ students may submit, each, a choice of a single college in C. If a student made no choice, her last choice is repeated. At the end of each day $t < 4$, for each college c:

- If the number of students who chose college c and have an exam grade at c higher than $\underline{z_c}$ at day t is smaller than q_c, let the cutoff value at c for period t, ζ_c^t, be $\zeta_c^t = \underline{z_c}$.
- Otherwise, ζ_c^t is set to be the q_c^{th} highest grade at c among those who chose c at t.
- The cutoff values $\zeta_{c_1}^t, \dots, \zeta_{c_m}^t$ are made public.

At the end of day $t = 4$, for each college c:

- The top q_c students who have an exam grade at c higher than $\underline{z_c}$ and chose c during day c are matched to college c.
- If the number of students who have an exam grade at c higher than $\underline{z_c}$ and chose c during day 4 is lower than q_c, all of them are matched to c.
- All students who chose c and were not matched to it will remain unmatched.
- All students who didn't apply to a college during all days will also remain unmatched.

Although the potential ability to know to which colleges a student may not be matched before submitting their final choice seems like an interesting property, in fact that is not the case in general, as noted in the following remarks.

Remark 1. Choices made during days $t = 1, \dots, 3$ may have no direct effect on the final outcome. As a result, students have no clear incentive to make choices before day 5.

Of course, if some student s makes a choice in a day $t^* < 4$ and doesn't make a choice in day 4, her choice during day t^* will be the one considered when generating the outcome at the end of day 4. However, the outcome would be the same if we kept other players' choices and s made her choice only at day 4. The fact that this results in no clear incentive for students to make choices before day 4 makes the information available by the end of day 3 about to which colleges student s could be matched to even less reliable .

[6] Different universities and programs could use different weights for the different parts of the exam. For example, Economics programs could give a higher weight to the math section of the exam, while Biology programs could give a higher weight to the biology section.

Remark 2. Cut-off values at some colleges may go down from one day to the next.

Since students may choose any college in any day, nothing prevents cut-off values at some colleges to go down from one period to the other. For example, consider a scenario in which college c has only one seat. Let student s, where $z_c(s) = 200$, be the only student choosing college c during day 3. The cut-off value for c made public at the end of day 3 is therefore $\zeta_c^4 = 200$. If s chooses a different college during day 4 and no other student chooses c, then $\zeta_c^4 = \underline{z}_c$. That is, some student s' whose grade at c is greater than $\underline{z}_c$ but lower than 200, cannot take the cut-off value at college c, even at the end of day 3, as an indication that she had no chance at being accepted there by the end of day 4.

If cutoff values go down from one day to another, then the use of those values as an information that guide students' applications away from schools at which they will not be accepted gets jeopardized. Moreover, if cutoff values go down at some program from day 3 to 4, a student who may have preferred to go to that program and get accepted by the end of day 4 will not do so.

4 Empirical evidence

In order to evaluate the empirical relevance of the shortcomings of the SISU mechanism identified in the previous section, we analyze data for the selection process that took place in January 2016. In that year, more than 228,000 seats in public universities were offered, and a total of more than 2,500,000 students paritipating. The average competition level, therefore, is of more than 10 candidates per seat.

The data consists of the cutoff values for each of the 25,686 options available to the students, for each of the four days in which students were able to make choices.[7] In Brazilian universities, students apply and may be accepted to specific programs in those universities, as opposed to joining the University as a whole. For example, a student must choose to apply, to the daytime Economics program at the Federal University of Rio de Janeiro, or to the nighttime Computer Science program at the same university. Although all programs use the national university entrance exam (ENEM), different programs may give different weights for different parts of the exam (essay, math, literature, etc) when ranking students.

We are interested in whether cutoff values decrease from one day to another, and if so by how much. As pointed out in section 3, cutoff values going down point to a failure in the SISU mechanism in providing information on the programs to which a student has no chance

[7] The process in 2016 differs from the rules used in 2010 in two aspects. First, the seats in each program offered by federal public universities are split into up to five different sets of seats, and eligibility to apply to each of those sets of seats differ across students (see Aygün and Bó [2013] for more details on the procedure). Secondly, instead of choosing only one program per day, students were able to specify first and second choices.

Finally, we note that although many programs have minimum exam grades in order for a student to be acceptable at that program, cutoff values are set to zero if the number of students acceptable who chose that program is below the capacity. These differences don't affect the fact that cutoff values being reduced from one period to another reduces the reliability of those values as an indication of programs at which a student has no chance getting into.

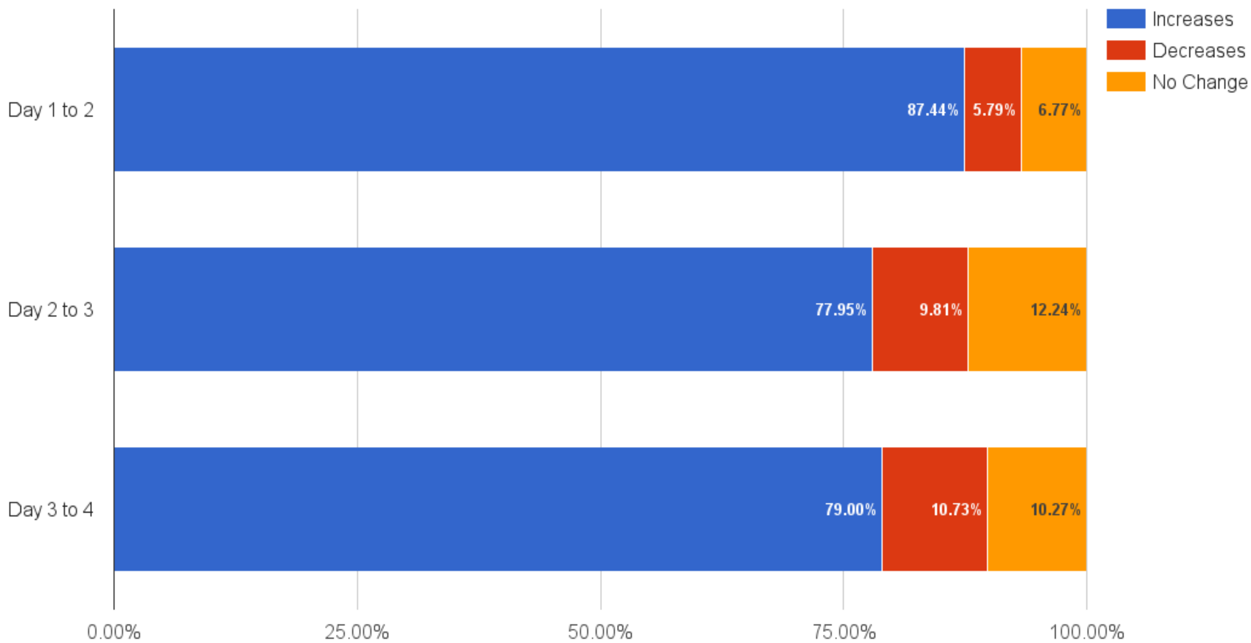


Figure 4.1: Proportion of programs at which cutoff values increased, decreased or didn't change from one day to the next

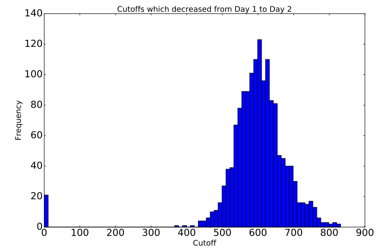
at being accepted, and moreover lead students not to choose programs that are preferred by those students and to which they would end up being accepted.

Figure 4 shows the proportion of programs available for the students at which the cutoffs increased, decreased or didn't change from one day to the next. There are some important facts to notice:

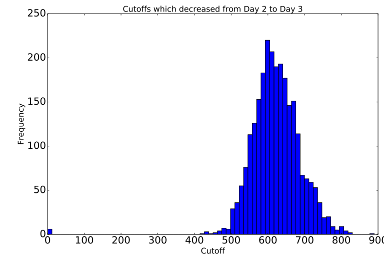
- The proportion of programs at which the cutoffs decreased is surprisingly high, on average 8.78% of the cutoffs,
- The proportion of programs at which the cutoffs decreased increased over time,
- More than 10% of the final cutoffs were lower than those informed to the students in the last day in which they made choices.

In all but five of the 25,686 programs available the cutoff value by the end of day 4 were zero. That is, in all but these five programs the number of students who chose those programs exceeded the number of seats. Figure 4.2 shows the histogram of the values of the cutoffs after they increased or decreased for each day. Although we can't say that the distributions of cutoffs which decreased and those which increased is not distinguishable, it seems clear that the decreases or increases are not clustered around different values of cutoffs.

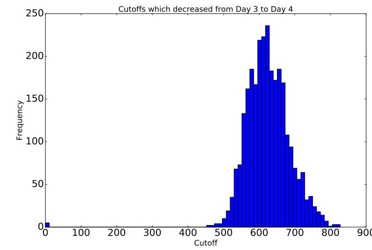
The next question is whether the changes in cutoff grades, when they decrease, are large enough to in fact affect students' beliefs and outcomes. If a cutoff decreases by a very small amount, for example, it may well be that no student could have been negatively affected by



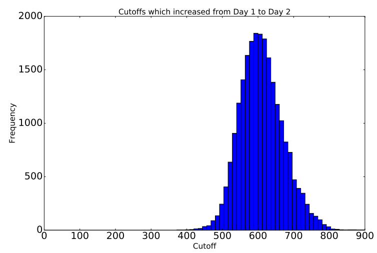
Decreased: Day 1 to 2



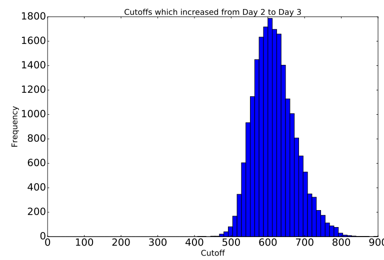
Decreased: Day 2 to 3



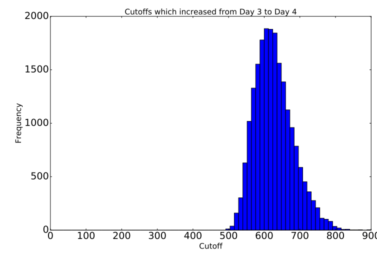
Decreased: Day 3 to 4



Increased: Day 1 to 2



Increased: Day 2 to 3



Increased: Day 3 to 4

Figure 4.2: Cutoff values after they were decreased/increased from the previous day

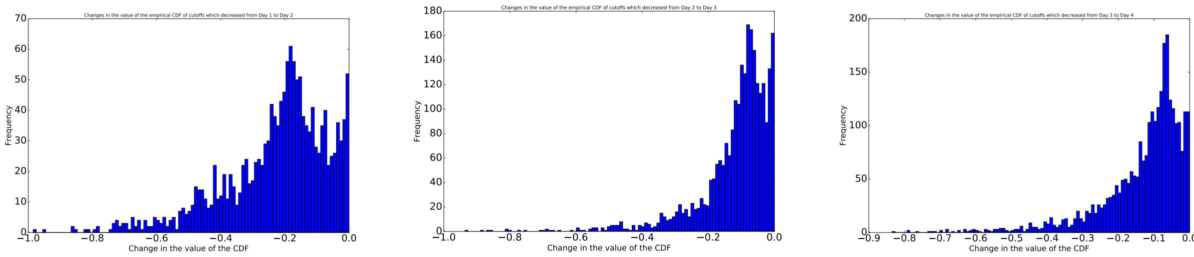
that change, since the number of students who become able to choose that program due to that decrease is small or even zero.

The measure that we use to evaluate the degree to which a cutoff value decreases is the change in the value of the empirical cumulative distribution function (CDF), at each program, from one day to the next. For example, say that the cutoff value at program p decreased from day 1 to day 2 from 550 to 500. If the value of the empirical CDF of all cutoffs in day 1 is of 0.3 in day 1 and 0.2 in day 2, then that means that 30% of the cutoff values were below the one for program $p$in day 1, but only 20% of them were below the one for program p in day 2.

Figure 4.3 shows the frequency of the changes in the value of the empirical CDF for each pair of consecutive days.[8] The graphs show that, although the largest changes take place from day 1 to day 2, in all cases the proportion of large changes in the ranking is quite significant. In fact, the percentage of programs where the change in the value of the CDF was lower than -0.2 was 46.87%, 14.61% and 19.39% for Day1/Day2, Day2/Day3 and Day3/Day4 respectively.

We can conclude, therefore, that the daily cutoff values which result from candidates interacting with the SISU mechanism fail to provide reliable information about the programs at which a student would not be accepted.

[8]All changes in the value of the CDFs were negative except for one, which had a change below 0.001 and was removed from the graphs for convenience.



Day 1 to 2 Day 2 to 3 Day 3 to 4

Figure 4.3: Change in the value of the empirical CDF for cutoff values that decreased

5 The Iterative Deferred Acceptance Mechanism

In this section we introduce the Iterative Deferred Acceptance Mechanism (IDAM). It consists essentially of the SISU mechanism with some important modifications, listed below. We will denote a student s as *tentatively accepted at college c by period t* if she chose college c at some period t^*, where $0 < t^* \leqslant t$ and for all t' such that $t^* \leqslant t' \leqslant t$, $\zeta_c^{t'} \leqslant z_c(s)$.

- **Commitment of choices**: Only students who are not tentatively accepted at some college in period $t-1$ are allowed to make a choice during period t. Moreover, when able to make a choice in period t, a student may only choose from colleges where the cutoff grade in period $t$is lower than her exam grade in that college.

- **Activity rule**: If a student s is allowed to make a choice in period t but doesn't, we consider that as choosing s (remaining unmatched) in that period.

- **Closing rule**: The mechanism ends after a period T in which every student is either tentatively accepted at some college or chose to remain unmatched at some period $t \leqslant T$ or when the number of periods reaches a predetermined number T^*.

Consider an exam-based college matching market $\langle S, C, q, P_S, P_C, \underline{Z}, z \rangle$ and maximum number of steps $T^* \in \mathbb{N}$. The mechanism proceeds as follows:

- **Step** $t = 0$: Let $\mathcal{L}^0 = S$, $S^0 = \varnothing$, and for for every $c \in C$, $\zeta_c^0 = \underline{z}_c$ and $\mu^0(c) = \varnothing$. Make public the values of $\zeta_{c_1}^0, \ldots, \zeta_{c_m}^0$.

- **Step** $0 < t \leqslant T^*$:

  - (a) Let $S^t \equiv \{s \in \mathcal{L}^{t-1} | \nexists c \in C : s \in \mu^{t-1}(c)\}$ and, for every $s \in S$, $\psi^t(s) \equiv \{c \in C : z_c(s) > \zeta_c^{t-1}\} \cup \{s\}$ if $s \in S^t$ and $\psi^t(s) = \varnothing$ otherwise.
  - (b) Request each student $s \in S^t$ to choose an element of $\psi^t(s)$. Let $\mathcal{L}^t$ be all students in $\mathcal{L}^{t-1}$ minus those who chose s (that is, to remain unmatched) and define, for each $c \in C$, $\mathcal{L}^t(c)$ be the set of students who chose c at this step.
  - (c) For each college c, let $\mu^{t*}(c) \equiv \mu^{t-1}(c) \cup \mathcal{L}^t(c)$.

  * If $|\mu^{t*}(c)| < q_c$, let $\zeta_c^t = \zeta_c^{t-1}$ and $\mu^t(c) = \mu^{t*}(c)$.
  * If $|\mu^{t*}(c)| = q_c$, let $\zeta_c^t = \min_{s \in \mu^{t*}(c)} \{z_c(s)\}$ and $\mu^t(c) = \mu^{t*}(c)$.
  * If $|\mu^{t*}(c)| > q_c$, let $\mu^t(c)$ contain the top q_c students with respect to z_c in $\mu^{t*}(c)$, and $\zeta_c^t = \min_{s \in \mu^t(c)} \{z_c(s)\}$.
- (d) Make the values of $\zeta_{c_1}^t, \ldots, \zeta_{c_m}^t$ public.
- (e) If for every $c \in C$ it is the case that $\mu^{t*}(c) = \mu^t(c)$, stop the procedure.

- The function μ^t, for the highest value reached of t, is the outcome of the mechanism. Denote by T the last step executed in the procedure.

The following lemma shows that regardless of the choices made by the students when interacting with the IDAM mechanism, the cutoff values at each college never go down.

Lemma 1. *(Cut-off grades never go down) For every* $0 \leqslant t \leqslant T$ *and* $c \in C$, $\zeta_c^t \geqslant \zeta_c^{t-1}$*. Moreover, if for every* $c \in C$ *it is the case that* $\zeta_c^{t^*+1} = \zeta_c^{t^*}$*, then* $T = t^* + 1$*.*

One of the consequences of Lemma 1 is that the IDAM mechanism always ends in finite time. We define, formally "straightforward behavior" [Roth and Sotomayor, 1992] when interacting with the IDAM mechanism:

Definition 1. A student $s \in S$ presents **straightforward behavior with respect to** P^* when interacting with the IDAM if, whenever there is a period in which she is requested to make a choice over a set $I \subseteq C \cup \{s\}$, she chooses $c^* \in I$, such that $\forall c' \in I : c^* R^* \, c'$, where R^* is the weak preference derived from P^*.

Proposition 1. *If all students present straightforward behavior with respect to the preference profile* P*, there is a finite number of steps* T *for which the outcome of the IDAM mechanism is the student-optimal stable matching with respect to* P*.*

Proof. When all students present straightforward behavior, the steps of the IDAM mechanism are the same as the steps of the algorithm presented in Dubins and Freedman [1981] if students, each time that they make a proposal, follow their preference ranking until accepted by some college. Therefore, the outcome will be the student-optimal stable matching. □

If the maximum number of steps, T^*, is not high enough, the outcome of the IDAM mechanism may not be stable when students follow the straightforward behavior. As shown in the lemma below, however, in that case all blocking pairs will involve a college and an unmatched student.

Lemma 2. *Let all students present straightforward behavior with respect to the preference profile* P *and* μ *be the matching produced by the IDAM mechanism. If a student* s *blocks* μ *with some college* c*, then* $\mu(s) = s$*.*

Proof. If the IDAM mechanism is run for enough periods, proposition 1 implies that μ is stable and therefore no student blocks μ with any college. Consider now the case in which the number of periods T^* is smaller than that, and suppose that there is a student s and a college c where $cP_s\mu(s)$, $\mu(s) = c'$ and student s and college c block μ. Since $\mu(s) = c'$, then at some period $t^* \leqslant T^*$, s chose college c'. Since s and c block μ, it must be that $\zeta_c^{T^*} < z_c(s)$. By lemma 1, $\zeta_c^{t^*} \leqslant \zeta_c^{T^*}$. Therefore, on period t^* both colleges c and c' were available to s but she chose c'. Contradiction with straightforward behavior with respect to P. □

6 Incentives and equilibria under the IDAM Mechanism

Although the outcome of the IDAM mechanism is the student-optimal stable matching when students present straightforward behavior, and differently from the GS-DA mechanism, [Dubins and Freedman, 1981, Roth, 1985], it is not the case here that students have a weakly dominant strategy in the game induced by the IDAM mechanism. In order to see this, we need first to formally define that game.

Fix a set of colleges C, with their capacities q and minimum scores $\underline{Z}$. The extensive game form G induced by the IDAM mechanism is a tuple (S, H, Φ, P, f) which consists of:

- A finite set of players $S = \{s_1, \ldots\}$.
- A finite set of actions $A = \{a_1, \ldots\}$.
- A set of finite histories H, which are sequence of actions, with the property that if $(a_i)_{i=1}^k \in H$, then for all $\ell < k$, $(a_i)_{i=1}^\ell \in H$. The null history, $h_\varnothing$ is also in H.
- At history h_0, nature draws the values of z and P from a joint distribution f, and each student s observes the realization of $z(s)$ and of P_s. The distribution f is common knowledge and may be degenerate.
- Let Z be the set of terminal histories, that is, if $h \in Z$ where $h = (a_i)_{i=1}^k$, then there is no $h' \in H$, where $h' = (a'_i)_{i=1}^\ell$ where $\ell > k$ and for all $i \leqslant k$, $a_i = a'_i$. Then $(a_i)_{i=1}^k \in Z \implies k \mod n = 0$.
- Φ is a player function. $\Phi : H \backslash Z \to S$. There exists an ordering of the players $(s_1, \ldots, s_n)$ such that, for all $h \in H$ such that $|h| \leqslant n$, $\Phi(h) = s_{|h|}$[9].
  - Let $(a_i)_{i=1}^k \in H$, where $k \geqslant 1$. If $(a_i)_{i=1}^{k+n} \in H$, then $\Phi\left((a_i)_{i=1}^k\right) = \Phi\left((a_i)_{i=1}^{k+n}\right)$[10].
- For each student s, $\mathcal{I}_s$ is a partition of $h : \Phi(h) = s$. Define $\zeta\left((a_i)_{i=1}^k\right)$ as the collection of lists of cutoff grades $((\zeta_c^0)_{c\in C}, (\zeta_c^1)_{c\in C}, \ldots)$ that result from the sequence of actions

[9]That is, the first n actions consist of player s_1 playing first, s_2 second and etc.

[10]This, combined with the previous item and the condition on terminal histories, implies that every player plays once every n actions.

in $(a_i)_{i=1}^{k-(k \mod n)}$. Define $H_\ell^t \equiv \left\{(a_i)_{i=1}^k \in H : k \mod n = \ell \text{ and } k \div n = t-1\right\}$[11], and let $h, h' \in H_\ell^t$. The histories $h = (a_i)_{i=1}^k$ and $h' = (a'_i)_{i=1}^k$ belong to the same member of the partition $\mathcal{I}_{s_\ell}$ if and only if[12] :

  - $|h| \mod n = |h'| \mod n$,
  - $\zeta(h) = \zeta(h')$,
  - $z(s_\ell|h) = z(s_\ell|h')$, that is, the realization of student s_ℓ's grades at the colleges are the same.
  - $a_i = a'_i$ for all i such that $i \mod n = \ell$ [13].

- $A(h)$ are the actions available at $h \in H$. For every $h_i \in H_\ell^t$, the set of actions depend on whether, given the history of actions until step t of the IDAM mechanism, student $s = \Phi(h_i)$ is offered a set of colleges to choose from, in which case $A(h_i) = \psi^t(s)$, or not, in which case we denote $A(h_i) = \{\diamondsuit\}$, where $\diamondsuit$ is simply a placeholder for an action when no action is requested from the student. We abuse notation and denote, for any $I_i \in \mathcal{I}_s$, $A(I_i)$ to be $A(h_i)$ for any $h_i \in I_i$ (remember that by definition all histories in I_i have the same set of actions associated to them).

- A (pure) strategy for player s is a function $\sigma_s(\cdot)$ that assigns an action in $A(I_i)$ to each information set $I_i \in \mathcal{I}_s$.

- The outcome function O assigns, to each strategy profile $\sigma = (\sigma_{s_1}, \ldots, \sigma_{s_n})$, the matching that results from following the unique terminal history that results from following those strategies in the IDAM mechanism, given the realization of z and P.

Since our solution concept will demand that students' strategies are rational at all possible information sets, we will need to consider how students' strategies act at each subgame. We first define a subgame:

Definition 2. A **subgame** of the game $G = (S, H, \Phi, P)$ at non-terminal history $h = (a_i)_{i=1}^k$, for $h \in H \backslash Z$ is a game $G|_h = (S|_h, H|_h, \Phi|_h, P|_h)$ where:

- $H|_h = \left\{h' = (a'_i)_{i=k}^l \text{ where } l \geqslant k \text{ and } (a_1, \ldots, a_{k-1}, a'_k, \ldots, a'_l) \in H\right\}$

- $S|_h = \{s \in S : \Phi(h') = s \text{ for some } h' \in H|_h \backslash Z\}$

- $\Phi|_h : H|_h \to S|_h$ such that for all $h' \in H|_h$, where $h' = (a'_i)_{i=k}^l$, $\Phi|_h(h') = \Phi(a_1, \ldots, a_{k-1}, a'_k, \ldots, a'_l)$

[11]That is, H_ℓ^t are all histories that student s_ℓ could reach after t steps of the mechanism.

[12]Notice that this game form has perfect recall.

[13]That is, two histories belong to the same member of the partition if the student's grades at the colleges are the same, the history of cutoffs was the same, and the actions taken by that player were the same. That is, if player s_ℓ has the same grades and the same experience in both histories.

- For each $s \in S|_h$, $P_s|_h$ satisfies, for all $h', h'' \in H|_h$, $h' \, P_s|_h \, h'' \iff (a_1, \ldots, a_{k-1}, a'_k, \ldots, a'_l) \, P_s \, (a_1, \ldots$ The weak preference $R_s|_h$ is defined accordingly.

Finally, let $\sigma|_h = (\sigma_{s_1}|_h, \ldots, \sigma_{s_n}|_h)$ be the strategy profile σ restricted to the subgame $G|_h$. We can define analogously a subgame in terms of an information set instead of a single history. We will consider situations in which students present straightforward behavior. Therefore, we can define a straightforward strategy accordingly:

Definition 3. A strategy σ_s of student $s \in S$ is **straightforward with respect to** P^* if for every t and $h^t_s \in H^t_s$:

$$\begin{cases} \sigma_s\left(h^t_s|z(s), P^*\right) = \max_{P*}\left(A\left(h^t_s\right)\right) & \text{if } A\left(h^t_s\right) \neq \{\diamondsuit\} \\ \sigma_s\left(h^t_s|z(s), P^*\right) = \diamondsuit & \text{otherwise} \end{cases}$$

The first question that we make is whether a student has a dominant strategy at the game induced by the IDAM mechanism. This is a natural question, since the mechanism itself resembles the deferred acceptance procedure and truth-telling is a weakly dominant strategy under that direct mechanism. As we show below, that is not the case under the IDAM mechanism.

Proposition 2. *A student may not have a weakly dominant strategy under the IDAM mechanism*

The reason why not following a straightforward strategy may be profitable is that, differently from the case with the deferred acceptance direct mechanism, an agent may influence others' actions by modifying the signals received by the other agents, in the form of different cutoff grades or rejections. So if, for example, a student has a strategy that depends on the cutoff grades in some different way, that fact could be exploited.

One interesting property of the IDAM mechanism, which is the driver of many of the theoretical results that will follow, is that altough the combination of strategies that students may use is much richer than that of straightforward strategies, the sequence of interactions that the students have with the mechanism cannot be distinguished from interactions that result from all students following straightforward strategies.

Lemma 3. *Let σ, and let be a strategy profile and h a history that results from that strategy profile. There is at least least one strategy profile σ^*, where every student follows a straightforward strategy with respect to some preference profile P^*, which also results in history h.*

The result in lemma 3 doesn't hold for the SISU mechanism, however.

Remark 3. There are sequences of actions that students may take under the SISU mechanism that cannot be produced by any profile of straightforward strategies.

To see why remark 3 is true, consider a student who is the most preferred student in colleges c_1 and c_2 and in period 1 chooses college c_1, in period 2 chooses c_2 and in period 3 chooses c_1 again. This sequence of actions is not possible under the IDAM mechanism, cannot be the result of a straightforward strategy (since in all periods both colleges are available to her) and can take place under the SISU mechanism.

Although not having a dominant strategy can be seen as an undesirable characteristic of the IDAM mechanism, when compared to the property of strategy-proofness, we will now show that students following straightforward strategies is, however, a robust equilibrium. First, we define our equilibrium concept.

Let A and B be two distributions over $C \cup \{s\}$. We denote by $\gtrdot_s$ the first-order stochastic dominance relation under P_s.

Definition 4. A strategy profile σ is an **ordinal perfect bayesian equilibrium (OPBE)** of a game $G = (S, H, \Phi, P, f)$ if for all $I_i \in \mathcal{I}_s$, every $s \in S|_{I_i}$, every assessment μ over $\mathcal{I}_s$ and strategy $\sigma'_s|_{I_i}$ for player s in the subgame $G|_{I_i}$:

$$O_\mu\left(\sigma_s|_h, \sigma_{-s}|_h\right) \;\gtrdot_s\; O_\mu\left(\sigma'_s|_h, \sigma_{-s}|_h\right)$$

The theorem below shows that students following straightforward strategies is an equilibrium.

Theorem 1. *Let σ^* be the strategy profile in which all strategies are straightforward. Then σ^* is a OPBE of the game induced by the IDAM mechanism.*

It is not the case, however, that every OPBE consists of every student following a straightforward strategy, as shown in the example below.

Example 1. Consider the following exam-based college matching market:

$$\begin{array}{ll} S = \{s_0, s_1\} & C = \{c_1, c_2\},\ q_i = 1 \\ P_{s_0} : c_1\ c_2 & P_{c_1} : s_0\ s_1 \\ P_{s_1} : c_1\ c_2 & P_{c_2} : s_0\ s_1 \end{array}$$

Let students have perfect beliefs (that is, beliefs are degenerate in the true values). Let student s_0 follow the straightforward strategy and s_1 follow an inverse strategy, consisting of choosing the least preferred acceptable college available at each step. This strategy profile is an OPBE.

Some further results.

Proposition 3. *Every stable matching is a Nash Equilibrium outcome of the game induced by the IDAM mechanism.*

Proof. Let μ be a stable matching. Make every student's strategy be applying in the first period to their match under μ, and not apply anywhere else if they are rejected afterwards. That is an equilibrium. □

Proposition 4. *Some Nash Equilibrium outcomes are not stable*

Proof. The example is based on a non-credible threat outside of the equilibrium path:

$$\begin{array}{ll} S = \{s_0, s_1, s_2, s_3\} & C = \{c_1, c_2, c_3, c_4\}, \ q_i = 1 \\ P_{s_0} : \boxed{c_1} \ c_2 \ c_3 \ c_4 & P_{c_1} : s_0 \ s_1 \ s_2 \ s_3 \\ P_{s_1} : c_1 \ \boxed{c_4} \ c_2 \ c_3 & P_{c_2} : s_0 \ s_1 \ s_3 \ s_2 \\ P_{s_2} : c_3 \ \boxed{c_2} \ c_1 \ c_4 & P_{c_3} : s_0 \ s_1 \ s_2 \ s_3 \\ P_{s_3} : c_2 \ \boxed{c_3} \ c_1 \ c_4 & P_{c_4} : s_0 \ s_1 \ s_2 \ s_3 \end{array}$$

Consider the following matching:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ s_0 & s_2 & s_3 & s_1 \end{pmatrix}$$

The matching μ is not stable, since (s_2, c_3) and (s_3, c_2) are blocking pairs. This outcome, however, is supported by the following strategy profile:

- σ_{s_0}: Apply to c_1 in step 1. If rejected, quit.
- σ_{s_1}: Apply to c_1 in step 1. If rejected:
  - and $z(c_3) = z(s_2)$ or $z(c_2) = 0$, apply to c_3. If then rejected, quit.
  - and $z(c_2) = z(s_3)$ or $z(c_3) = 0$, apply to c_2. If then rejected, quit.
  - otherwise, apply to c_4. If then rejected, quit.
- σ_{s_2}: Apply to c_2 in step 1. If then rejected, quit.
- σ_{s_3}: Apply to c_3 in step 1. If then rejected, quit.

Student s_0 gets her top choice, so she wouldn't deviate. Given student s_0's strategy and the fact that she has top priority in c_1, student s_1 would not be able to be matched to c_1 and therefore has no profitable deviation. Consider now student s_2. Any profitable deviation strategy must apply to some school at step 1, otherwise she will remain unmatched. Moreover, any strategy that starts applying to c_2 will not change her outcome. We must then check all other possibilities:

- Apply to c_1 in the first step. Then s_2 is rejected from c_1 at step 1. Since $z(c_2) = 0$, student s_1 will then apply to c_3 and remain matched there. Therefore, the only remaining options for s_2 will be to quit or to apply to c_2 or c_4. In both cases there is no improvement over μ.
- Apply to c_3 in the first step. Then s_2 is tentatively accepted at c_3. Since $z(c_3) = z(s_2)$, however, in step 2 student s_1 will apply to c_3, leading to the rejection of s_2. Again the only remaining options for s_2 will be to quit or to apply to c_2 or c_4. In both cases there is no improvement over μ.

- Apply to c_4 in the first step. Since $z(c_2) = 0$, student s_1 will then apply to c_3 and remain matched there. Student s_2 will not be rejected from c_4 and therefore will remain matched there. Since $c_2 P_{s_2} c_4$, that is not a profitable deviation.

The same analysis for s_3 will show that she also has no profitable deviation, and therefore μ is an equilibrium outcome for the game induced by the iterative mechanism. □

One important fact to notice is that the schools' priorities in the example used above has an *Ergin-acyclic priority structure*. Haeringer and Klijn (2009) show that when the priority structure is Ergin-acyclic, the set of outcomes of the game induced by the SPDA mechanism equals the set of stable matchings. We can, therefore, conclude the corollary below.

Corollary 1. *The set of Nash equilibrium outcomes for the IDAM mechanism is not equal to the set of equilibrium outcomes for the SPDA.*

7 Manipulations via cutoffs

Other than the fact that, under the SISU mechanism, the cutoff values do not represent a reliable information regarding the chances that a student has at being accepted into a college, that mechanism is also subject to what we denote by *manipulation via cutoffs*. A manipulation via cutoffs occurs when a group of students artificially increases the cutoff values of some college, as a way to prevent applications from other students, and then in the last period vacating those seats so that students with lower exam grade take their place. The example below shows how manipulations via cutoffs can happen.

Example 2 (Manipulation via cutoffs)**.** Consider the set of students $S = \{s_0, s_1, s_2, s_3\}$ and of colleges $C = \{c_1, c_2, c_3\}$, each with capacity $q_i = 1$ and minimum score zero. Students' preferences are as follows:

$$P_{s_0} : c_1\ c_2\ c_3$$
$$P_{s_1} : c_1\ c_2\ c_3$$
$$P_{s_2} : c_1\ c_2\ c_3$$
$$P_{s_3} : c_2\ c_1\ c_3$$

Students' exam grades at the colleges are as follows:

	c_1	c_2	c_3
s_0	100	100	100
s_1	200	200	200
s_2	300	300	300
s_3	400	400	400

Suppose that the SISU mechanism is going to be used, and students present straightforward behavior. The cut-off values, at the end of each period would then be as follows (remember that the cut-offs at $t = 4$ represent the final allocation cutoffs):

	c_1	c_2	c_3
$t=1$	300	400	0
$t=2,3,4$	300	400	200

The matching produced, therefore, will be μ:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 & \varnothing \\ s_2 & s_3 & s_1 & s_0 \end{pmatrix}$$

Suppose, however, that students s_0 and s_3 modify their behavior, and act instead as follows:

- During $t = 1, 2, 3$, student s_0 chooses college c_3 and student s_3 chooses college c_1.
- In period $t = 4$, student s_0 chooses college c_1 and student s_3 chooses college c_2.

Assuming that the other students present straightforward behavior, the cut-off values, at the end of each period, would be as follows:

	c_1	c_2	c_3
$t=1$	400	$---$	100
$t=2$	400	300	100
$t=3$	400	300	200
$t=4$	100	400	200

The matching produced will be μ':

$$\mu' = \begin{pmatrix} c_1 & c_2 & c_3 & \varnothing \\ s_0 & s_3 & s_1 & s_2 \end{pmatrix}$$

Student s_0 is signifficantly better off under μ' than under μ, while s_3 is matched to the same college in both cases.

Manipulations via cutoffs consists, in other words, of a set of students S^H "holding" seats in colleges and "releasing" them so that a set of students S^T can take them in the last period. In order for these type of manipulations to be successful, some conditions need to be satisfied.

First of all, the set of students S^H needs to be large enough when compared to the capacity of the college, and their exam grades in that college must be high enough. That the exam grades of the students in S^H should be high is If the number of students in S^H is low when compared to the capacity of the college, the effect of them choosing that college in the value of the cutoff will be much less noticeable. To see that, consider the case in which, at a certain period, there are 100 students choosing college c, which has a capacity of 10 students, and for simplicity assume that those students' scores fill the range $\{1, 2, \ldots, 100\}$ (that is, one student has a score 1, one has a score 2, etc). Then, given those choices, the students who will be tentatively accepted will be those with scores 91 to 100, and therefore the cutoff value will be 91. Suppose that S^H has 5 students, with exam grades $\{300, 301, 302, 303, 304\}$. These are, of course, signifficantly larger than the other students. If all of them choose college c in addition to the 100 students, all of them will be tentatively accepted in that period, but the change in the cutoff value will not be as signifficant: it will change from 91 to 96. If the capacity of the college was five, the change in the cuttof would be, instead, from 96 to 300. It's not necessarily the case that the number of students in S^H has to be equal to the college's capacity for the change in cutoff to be significant. Consider the case in which the exam scores of the 100 students choosing c are, instead, $\{252, 251, 250, 100, 99, 98, \ldots 4\}$, and the capacity still be five. The cutoff value for college c would be 99 in that period. If S^H has only 2 students, with exam grades $\{300, 301\}$, them choosing c would lead the cutoff grade at c to change from 99 to 250, instead.

Second, the other students have to respond in a straightforward way to the cutoff values in the last period. This can be considered a reasonably mild requirement. It doesn't require that the other students follow a straightforward strategy in all periods, but only that they wouldn't choose, in the last period, a college where the cutoff value is above their grade in that college.

One may wonder how realistic the first condition is. After all, colleges typically accept thousands of students every year, and a coalition of hundreds of high-achieving students performing these potentially risky manipulations doesn't seem realistic. In many countries (including Brazil and China), however, students apply directly to specific programs in the universities, so even though the universities as a whole accept hundreds or thousands of students, the number of seats at each program is often below 100, and many times lower than 30 or 20. Moreover, even those seats are often sub-divided. In China, the seats in each program are separated between seats reserved for candidates from specific provinces. In Brazil, federal universities partition the seats in the programs into five sets of seats, reserved for different combinations of ethnic and income characteristics. Finally, universities sometimes offer only a subset of the total number of seats in a program through the centralized matching process. In fact, *the median number of seats offered in each option available during the January 2016 selection process in Brazil, where more than 228,000 seats in public universities were offered, was five.*

There is evidence that this type of manipulation takes place in real life. In the chinese province of Inner Mongolia, a mechanism which has some similarities to the SISU mechanism is used to match students to programs in universities. While the mechanism itself has significant differences, it is also vulnerable to manipulation via cutoffs. This fact seems to be exploited by students, as documented by China News:[14]

> (...) in fact, since 2008, the clearinghouse found that some high scored students applied to a college with lower cutoff score. For example, their score allows them to go to PKU or Tshinghua, but they chose Beijing Polytech first. On the other hand, some other students, from the same high school often, applied to college that their score would not allow them to go initially.... system shows that their rank is below the capacity — so they can't be admitted under usual terms — however they do not revise their choices.

Even more remarkably, there seems to be evidence that high schools are coordinating students' actions:

[14]Source (in Chinese): http://edu.people.com.cn/n/2014/0904/c1053-25604075.html

> (...) the clearing house noticed that, 2 or 3 min before the deadline, the ranking of students in the system is changing – this is the evidence that high schools are organizing their own high scored students to occupy seats for low scored students

Differently from the SISU mechanism, the IDAM doesn't have this characteristic:

Remark 4. The IDAM mechanism is not manipulable via cutoffs.

It's easy to see why that's the case. In order for manipulations via cutoff to work, it is necessary for cutoff values to increase before the final allocation is determined, and for the final cutoff values (that is, the allocation cutoffs) to be lower. By Lemma 1, this cannot happen under the IDAM mechanism.

8 Convergence speed and stability

In this section we consider two related questions. As we saw in the description of the IDAM mechanism, the number of steps until it reaches the Student Optimal Stable Matching when students follow the straightforward strategy depends on preferences and exam grades. One question then is how many steps it takes for that result to be produced.

A second question is how "far" from a stable matching the outcome will be if the IDAM mechanism is run for a number of periods smaller than that necessary to produce the Student Optimal Stable Matching but students still follow the straightforward strategy. The measure of distance from a stable matching that we use is the number of individuals involved in blocking pairs.

8.1 Analytical results

For the results below, we consider exam-based college matching markets where the set of students and colleges can be partitioned as $S = \left\{S^1 \cup S^2 \cup \cdots \cup S^k\right\}$ and $C = \left\{C^1 \cup C^2 \cup \cdots \cup C^k\right\}$, where $\sum_{c \in C^i} q_c \leqslant |S^i|$ and colleges at C^i prefer students at S^i to those not in S^i. This is consistent with situations such as colleges (or schools) preferring students who live in its district, or math/literature exams where students are good at either math or at literature. Notice, however, that when $k = 1$ this definition accomodates any market in which the number of seats in colleges doesn't exceeds the number of students. We also assume in this section that students follow the straightforward strategy.

Proposition 5. *If for every $i \in \{1, \ldots, k\}$, $c, c' \in C^i$ and $s, s' \in S^i$ it is the case that $sP_cs' \iff sP_{c'}s'$ and moreover for all $s \in S^i$, $c \in C^i$ and $c \notin C^i$ it is the case that cP_sc', then:*

1. The maximum number of steps until stability is $\max_i \{|C^i|\}$.

2. If the IDAM mechanism run for $T < \max_i \{|C^i|\}$ steps, the maximum number of individuals involved in blocking pairs is $n - \sum_{j=1}^{k} \sum_{i=1}^{T} q_i^j$, where for each j, $q_1^j \leqslant q_2^j \leqslant \cdots \leqslant q_{|C^j|}^j$ is the ordering of the capacities of the schools in C^j.

Proof. Since students follow straightforward strategies, a student $s \in S^i$ will only apply to colleges that are not in C^i if the cutoffs at all colleges in C^i are above her exam grade. Moreover, since for every i the number of students who prefer any college in C^i to any college not in C^i is at least as big as the overall number of seats in these colleges, by the end of the execution of the IDAM mechanism all seats in those colleges will be occupied by students in S^i, and students in S^i who are not matched to colleges in C^i will be left unmatched (even though some of them may be tentatively accepted at some period during the execution of the mechanism). From the perspective of a student in S^i, therefore, a seat in a college in C^i which is being occupied by a student not in S^i is equivalent to an empty seat.

Consider any $i \in \{1, \ldots, k\}$ and let $q_1^i \leqslant q_2^i \leqslant \cdots \leqslant q_{|C^i|}^i$ be the ordered capacities of the colleges in C^i. We will denote by $\left\{S_1^i, S_2^i, \ldots, S_{|C^i|}^i, S_-^i\right\}$ the partitioning of the students in S^i where S_1^i are the top q_1^i students in S^i in colleges C^i's preferences, S_2^i are the top q_2^i students after those in S_1^i in colleges C^i's preferences, etc, and S_-^i are the students in S^i below the top $\sum_{j=1}^{|C^i|} q_j^i$ students. By proposition 1, when students follow straightforward strategies the final outcome of the IDAM mechanism is the student-optimal stable matching, and by lemma 2, at any period in which there are blockings, those involve students who are not tentatively matched to any college. The number of students who are involved in a block is therefore maximized when the number of students tentatively accepted to a college any period is minimal.

Consider now period $t = 1$. Since every college in C^i has at least q_1^i seats, every student in S_1^i will be accepted at any college in that period. There is one case in which all other students will be rejected, though: if all students in S^i choose the same college with capacity q_1^i in period $t = 1$. In that case, $|S^i| - q_1^i$ students in S^i will be tentatively unmatched by the end of period 1, and therefore if the IDAM mechanism runs for just one period, that is the maximum number of students in S^i who will be involved in blocking pairs. The same argument will follow at $t = 2$: given that the students in S_1^i are all matched to a school with capacity q_1^i, the number of students who are tentatively unmatched by period $t = 2$ is maximal when all the remaining students in S^i choose a college with capacity q_2^i.

If we consider all the colleges and students, this process will take place in parallell at each element of $S = \left\{S^1 \cup S^2 \cup \cdots \cup S^k\right\}$ and $C = \left\{C^1 \cup C^2 \cup \cdots \cup C^k\right\}$. That is, by the end of period 1, the maximum number of students involved in blocks in S^1 is $|S^1| - q_1^1$, in S^2 is $|S^2| - q_1^2$, etc. The result extends, therefore, to a maximum of $n - \sum_{j=1}^{k} \sum_{i=1}^{T} q_i^j$ students involved in blocks.

Finally, if we consider the maximum number of steps that it takes until the student-optimal stable matching is produced, we can ask about which preferences from the students minimize the number of students who are matched to their final allocation at each step. That is, by minimizing the number of students matched to their final allocation we allow for the maximum number of students who can still make choices. Here it is easy to see that the preferences considered above, in which all students apply to the colleges in order of increasing capacity, is also the one that at each step matches the minimal number of students to their final allocation. The overall process will in that case end when the last set in $\left\{S^1_{|C^1|}, S^2_{|C^2|}, \ldots, S^k_{|C^k|}\right\}$ is matched to their final allocation. That will be, therefore, the one with the largest number of colleges. Therefore, the maximum number of steps is $\max_i \{|C^i|\}$. □

The configuration of preferences used in proposition 5 is consistent with scenarios in which the top preferences are partitioned mutually between students and colleges, and colleges share the selection criteria among their top students. One example would be a college admissions program that is based on national exams consisting in questions on different subjects and college programs that rank the students based on their grades in those different subjects. The stronger assumption in this case is that the partition is such that students are good at only one of the subjects. For example, if the partitioning of college programs is between Medical Sciences, STEM and Humanities, a student who is good at humanities isn't at STEM or medical subjects.

For the case of common preferences between all colleges, the result doesn't have to rely on some assumption on students' strategy.

Corollary 2. *When priorities are common across colleges and the IDAM mechanism runs for $T < m$ steps, the maximum number of individuals involved in blocking pairs is $n - \sum_{i=1}^{T} q_i$, where the capacities of the colleges in C are reordered such that $q_1 \leqslant q_2 \leqslant \cdots \leqslant q_m$.*

Proposition 6. *If for every $i \in \{1, \ldots, k\}$, $c, c' \in C^i$ and $s, s' \in S^i$ it is the case that $cP_sc' \iff cP_{s'}c'$, the maximum number of steps until stability is $\max_i \{|C^i|\}$.*

Proof. Because of the priorities, it doesn't really matter if a student at some point is tentatively accepted at some school outside of her partition, so it is as if those seats were still available to the students from that partition. With that, it then becomes simply a problem in which students all apply to the top, then second, then etc school. So the maximum number of steps until stability will be the maximum number of steps in one of those partitions, which is the maximum number of schools in a partition. □

Proposition 7. *Consider the configuration in the last proposition and let the mechanism run for $T < \max_i \{|C^i|\}$ steps and players follow the straightforward strategy. Then the maximum number of individuals involved in blocking pairs is $\max\left\{0, n - \sum_{j=1}^{k} \sum_{i=1}^{T} q_i^j\right\}$, where for each j, $q_1^j \leqslant q_2^j \leqslant \cdots \leqslant q^j_{|C^j|}$ is the ordering of the schools in C^j.*

Corollary 3. *Under common preferences, the maximum number of steps until stability is m. Also under common preferences, the maximum number of individuals involved in blocking pairs if the IDAM mechanism runs for T periods is $\max\left\{0, n - \sum_{i=1}^{T} q_i\right\}$, where q_i uses the ordering of capacities above., where q_i uses the ordering of capacities above.*

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A Appendix

A.1 The SISU 2016 procedure

The procedure used during the 2016 version of the SISU selection process has two differences when comparing with the 2010 version, described in section 3: the presence of affirmative action quotas and the fact that students were able to, instead of choosing only one program in each period, choose a first and a second option.

A.1.1 Affirmative action quotas

Due to a federal affirmative action law, many of the programs offered in the SISU have their seats split into five sets: Q_{MI}, Q_{mI}, Q_{Mi}, Q_{mi} and Q_{-}, with capacities q_{MI}, q_{mI}, q_{Mi}, q_{mi} and q_{-}, respectively. A student may apply to only one among these in a program, but eligibility to apply to each of those depend on her status as a low-income student, as a student who is a racial minority and whether she studied in a public high-school:

- Only candidates who studied in a public high-school, belong to a racial minority and provide evidence of belonging to a low-income household may apply to a seats in Q_{MI},
- Only candidates who studied in a public high-school and provide evidence of belonging to a low-income household may apply to a seats in Q_{mI},
- Only candidates who studied in a public high-school and belong to a racial minority may apply to a seats in Q_{Mi},
- Only candidates who studied in a public high-school may apply to a seats in Q_{mi},
- Any candidate may apply to seats in Q_{-}.

The cutoff values that are published by the SISU mechanism are calculated, for each set of seats in each program, exactly the same as described in section 3. The cutoff for the seats in Q_{mI} in period t, which we will here denote by ζ^t_{mI}, is the q_{mI}th highest exam grade in the program among those who applied to the set of seats Q_{mI}. Notice that a student who who studied in a public high-school and provide evidence of belonging to a low-income household, but doesn't belong to a racial minority, may apply to the seats in Q_{mI} but could instead choose to apply to the seats in Q_{mi} or Q_{-}. That student would not eligible to apply to the seats in Q_{MI} or Q_{Mi}, however.

Consider now a student s, who prefers program c_1 to c_2. Depending on her status as a low-income student, as a student who is a racial minority and whether she studied in a public high-school, she may be eligible to apply to some subset of the five sets of seats available in each program. For a student who is eligible to apply to seats in Q_{MI}, for example, all the cutoffs in the sets of seats in a program could be used as informative on whether that student has any chance getting into that program. For example, if the cutoffs at all sets of seats in a program are higher than that student's exam grade, then this would indicate that the student cannot get into that program. If, on the other hand, any of those cutoffs were lower than that student's grade, then she could apply to those seats.

If we assume that the students will always choose to apply to any set of seats to which they are eligible to, then there are some instances in which a reduction in a cutoff doesn't affect some students' ability to be accepted into a program. For example, consider a case in which $\zeta^t_{MI} > \zeta^{t+1}_{MI} > \zeta^t_{mI} = \zeta^{t+1}_{mI}$. The value of the cutoff for the seats in Q_{MI} decreased, but notice, however, that this change doesn't make any student who is eligible to apply to seats in Q_{MI} and would not be accepted at the program in period t accepted at period $t+1$. This is the case because if the exam grade for that student is between ζ^{t+1}_{MI} and ζ^t_{MI}, then that student could be accepted in both periods by applying to the seats in Q_{mI}, where her exam grade is high enough to be accepted in both periods. This observation generalizes as follows:

- Students eligible to apply to seats in Q_{MI} may be affected by a decrease in cutoff only if $\min\left\{\zeta^{t+1}_{MI}, \zeta^{t+1}_{mI}, \zeta^{t+1}_{Mi}, \zeta^{t+1}_{mi}, \zeta^{t+1}_{-}\right\} < \min\left\{\zeta^t_{MI}, \zeta^t_{mI}, \zeta^t_{Mi}, \zeta^t_{mi}, \zeta^t_{-}\right\}$,
- Students eligible to apply to seats in Q_{mI} may be affected by a decrease in cutoff only if $\min\left\{\zeta^{t+1}_{mI}, \zeta^{t+1}_{mi}, \zeta^{t+1}_{-}\right\} < \min\left\{\zeta^t_{mI}, \zeta^t_{mi}, \zeta^t_{-}\right\}$,
- Students eligible to apply to seats in Q_{Mi} may be affected by a decrease in cutoff only if $\min\left\{\zeta^{t+1}_{Mi}, \zeta^{t+1}_{mi}, \zeta^{t+1}_{-}\right\} < \min\left\{\zeta^t_{Mi}, \zeta^t_{mi}, \zeta^t_{-}\right\}$,
- Students eligible to apply to seats in Q_{mi} may be affected by a decrease in cutoff only if $\min\left\{\zeta^{t+1}_{mi}, \zeta^{t+1}_{-}\right\} < \min\left\{\zeta^t_{mi}, \zeta^t_{-}\right\}$,
- Students eligible to apply to seats in Q_{-} may be affected by a decrease in cutoff only if $\zeta^{t+1}_{-} < \zeta^t_{-}$.

Our empirical analysis, therefore, only consider that subset of reductions in cutoffs.

A.1.2 First and second choice

Differently from the procedure used in 2010, in 2016, in each period, students were asked to indicate a first and second choice among the available sets of programs[15]. The cutoff values calculated at the end of each period were based on the execution of the following algorithm:

[15] More precisely, the students could choose two programs, one set of seats in each. For example, the first choice could be the mi seats in program c_1 and the second choice would be the Mi seats in program c_2. Since for the purpose of the analysis of the effect of the availability of these two choices that fact isn't relevant, we will consider simply the students' choices over programs.

1. Consider all students' first and second choice as being their preferences over C. That is, if student s submitted c_1 as her first choice and c_2 as her second choice, her full preference are considered as being $c_1 P_s c_2 P_s s$.

2. Using programs' preferences responsive to exam scores, let μ_t^C be the outcome of the **college-proposing deferred acceptance** algorithm.

3. For each program c such that $\left|\mu_t^C(c)\right| < q_c$, let the cutoff for that period ζ_c^t be $\underline{z}_c$, that is, the minimum exam grade for acceptance at c.

4. For each program c such that $\left|\mu_t^C(c)\right| = q_c$, let the cutoff for that period ζ_c^t be $\min_{s\in\mu_t^C(c)} z_c(s)$, that is, the lowest exam grade at c among those in $\mu_t^C(c)$.

It is easy to see that the procedure in 2010 is the same as the one described above with the difference that instead of using two choices as being the student's preference one considers instead a preference in which a student considers only one college acceptable.

Since students can only submit two colleges, it's still the case that the students have to consider carefully which one to submit, since she may end up not matched to any school. What is left to evaluate, however, is whether a student would always be able to be accepted at a college which have a cutoff value lower than her grade at that school.

Consider an exam-based college matching market $\langle S, C, q, P_S, P_C, \underline{Z}, z\rangle$, let $P_S^* = (P_s^*, P_{-s})$ be a preference profile where P_s^* differs from P_s in that college $c^* \in C$ is deemed as unacceptable for s and let μ^C be the college-optimal stable matching for the exam-based college matching market $\langle S, C, q, P_S^*, P_C, \underline{Z}, z\rangle$. Suppose that μ^C is blocked by s and c^*. We want to show that the college-optimal stable matching μ^{C*} for the market $\langle S, C, q, P_S^{**}, P_C, \underline{Z}, z\rangle$, where $P_S^{**} = (P_s^{**}, P_{-s})$ and P_s^{**} is the preference for s in which only college c^* is acceptable, is such that $\mu^{C*}(s) = c^*$.

To see that this is the case, consider the college proposing deferred acceptance procedure for the market $\langle S, C, q, P_S^*, P_C, \underline{Z}, z\rangle$ and college c^*. At each step of the deferred acceptance procedure, college c^* proposes to the top q_{c^*} students, who didn't yet reject c^*, with respect to z_{c^*}. Let T be the number of steps in the algorithm until the matching μ^C is produced. The set $\mu^C(c^*)$ consists of the top students in S, with respect to z_{c^*}, who didn't reject c^* at some step. Since μ^C is blocked by s and c^*, either $\left|\mu^C(c^*)\right| < q_{c^*}$ or there is a student $s' \in \mu^C(c^*)$ such that $z_{c^*}(s) > z_{c^*}(s')$.

Suppose that the statement is false, that is, that the college-optimal stable matching μ^{C*} under the market $\langle S, C, q, P_S^{**}, P_C, \underline{Z}, z\rangle$ is such that $s \notin \mu^{C*}(c^*)$. Since offers made by colleges during the college-proposing DA are not withdrawn, it must be that college c^* doesn't make an offer to s, implying that college c^* has under μ^{C*} a more preferred set of students than under μ^C. Now consider the deferred acceptance steps in $\langle S, C, q, P_S^{**}, P_C, \underline{Z}, z\rangle$ compared to those in $\langle S, C, q, P_S^*, P_C, \underline{Z}, z\rangle$, and consider the first step at which the proposals and rejections are different. These differences can only come if student s rejects colleges considered acceptable under P_s^* but unacceptable under P_s^{**}. That is, they come from the fact that student s will reject a set of colleges weakly larger than under P_s^*. This implies that those colleges will make offers that are further down in their preference. Those further rejections will also, at each step, weakly increase the set of students who reject offers from colleges. That is, in terms of colleges' preferences, they are weakly worse off. But this contradicts the assumption that $s \notin \mu^{C*}(c^*)$ and s blocking μ^C with c^*, since it would be necessary that c^* obtains a more preferred set of students under μ^{C*} in order not to make an offer to s.

Therefore, the result above shows that if a program has a cutoff value that is below a student's grade in that program, the student would be able to get accepted at that school by modifying her preference, putting that program as her top choice. Similarly, if the cutoff is higher than that student's grade, that student would never receive an offer from that school during the deferred acceptance procedure.

We can conclude, therefore, that under the the problems associated with the possibility of reduction in the cutoff values described in section 3 are also present in the 2016 version of the SISU mechanism.

A.2 Proofs

Lemma 1

Proof. We prove by induction. Let $t = 1$. From step $t = 0$, for every $c \in C$, $\zeta_c^0 = \underline{z}_c$. For each college $c \in C$, we have three cases:

- $|\mu^{1*}(c)| < q_c$, in which case $\zeta_c^1 = \zeta_c^0$
- $|\mu^{1*}(c)| = q_c$. Since $\mu^0(c) = \varnothing$, $\mathcal{L}^1(c)$ consists of the set of students who chose college c at step $t = 1$. Given the definition of $\psi^1(s)$, for every student $s \in \mathcal{L}^1(c)$, $z_c(s) > \zeta_c^0$. Therefore, $\zeta_c^1 = \min_{s \in \mu^{1*}(c)} \{z_c(s)\} > \zeta_c^0$.
- $|\mu^{1*}(c)| > q_c$. Since every student in $\mu^1(c)$ is also in $\mathcal{L}^1(c)$, $\zeta_c^1 = \min_{s \in \mu^1(c)} \{z_c(s)\} > \zeta_c^0$.

Now assume that for every $t = 0, \ldots, k$, where $k < T$ and for every $c \in C$, it is the case that $\zeta_c^t \geqslant \zeta_c^{t-1}$ and consider the step $k + 1$. For each college $c \in C$, we have three cases:

- $\left|\mu^{k+1*}(c)\right| < q_c$, in which case $\zeta_c^{k+1} = \zeta_c^k = \cdots = \zeta_c^0$[16].
- $\left|\mu^{k+1*}(c)\right| = q_c$. We have two cases. If $\left|\mu^k(c)\right| < q_c$, then $\zeta_c^k = \zeta_c^0$. Note that since $\left|\mu^{k+1*}(c)\right| = q_c$, $\left|\mathcal{L}^{k+1}(c)\right| > 0$. By definition, for every student $s \in \mathcal{L}^{k+1}(c)$, $z_c(s) > \zeta_c^k = \zeta_c^0$. Since $\mu^{k+1}(c) = \mu^k(c) \cup \mathcal{L}^{k+1}(c)$, $\min_{s \in \mu^{k+1}(c)} \{z_c(s)\} \geqslant \min_{s \in \mathcal{L}^{k+1}(c)} \{z_c(s)\} > \zeta_c^0$. Therefore $\zeta_c^{k+1} > \zeta_c^0 = \zeta_c^k$. If, on the other hand, $\left|\mu^k(c)\right| = q_c$, since $\left|\mu^{k+1*}(c)\right| = q_c$ it must be the case that $\left|\mathcal{L}^{k+1}(c)\right| = 0$, and therefore $\min_{s \in \mu^{k+1}(c)} \{z_c(s)\} = \min_{s \in \mu^k(c)} \{z_c(s)\} = \zeta_c^k$, and therefore $\zeta_c^{k+1} = \zeta_c^k$.

[16] Notice that $\left|\mu^{k+1*}(c)\right| < q_c$ together with the fact that students who are tentatively matched cannot change their submission implies that the cutoff grade will only raise from ζ_c^0 once the number of tentatively accepted students reaches q_c.

- $\left|\mu^{k+1*}(c)\right| > q_c$. Since $\left|\mu^{k+1*}(c)\right| > q_c$ and $\mu^{k+1*}(c) \equiv \mu^k(c) \cup \mathcal{L}^{k+1}(c)$, $\left|\mathcal{L}^{k+1}(c)\right| > 0$. If $\left|\mu^k(c)\right| < q_c$, $\zeta_c^k = \zeta_c^0$ and since for every student $s \in \mathcal{L}^{k+1}(c)$, $z_c(s) > \zeta_c^k = \zeta_c^0$, $\zeta_c^{k+1} = \min_{s \in \mu^{k+1}(c)} \{z_c(s)\} > \zeta_c^0$. Otherwise if $\left|\mu^k(c)\right| \geqslant q_c$, $\zeta_c^k = \min_{s \in \mu^k(c)} \{z_c(s)\}$. That is, there are q_c students in $\mu^k(c)$ with exam grade at c greater than or equal to ζ_c^k. Moreover, by definition, for every $s \in \mathcal{L}^{k+1}(c)$, $z_c(s) > \zeta_c^k$. That is, there is at least one student in $\mathcal{L}^{k+1}(c)$ and all those students have an exam grade at c higher than the student in $\mu^k(c)$ who has the lowest exam grade at that college. Therefore, in $\mu^k(c) \cup \mathcal{L}^{k+1}(c)$ there are at least q_c students with exam grade at c strictly greater than ζ_c^k, and as a consequence the q_c^{th} highest exam grade in $\mu^{k+1*}(c)$ is strictly greater than ζ_c^k. Therefore, $\zeta_c^{k+1} = \min_{s \in \mu^{k+1}(c)} \{z_c(s)\} > \zeta_c^k$.

Now, for the second statement in the lemma, fix a $t \geqslant 0$ and suppose that for every $c \in C$ it is the case that $\zeta_c^{t+1} = \zeta_c^t$. We can use the two parts of the proof by induction above to conclude that there are two scenarios compatible with that assumption:

- $t = 0$ and for all $c \in C$, $|\mu^{1*}(c)| < q_c$. In this case, the definition of step 1(c) establishes that, for each c, $\mu^1(c) = \mu^{1*}(c)$. But then step 1(d) implies that the procedure will stop at step $t+1$.

- $t > 0$ and for every $c \in C$, either $|\mu^{t+1*}(c)| < q_c$ or $|\mu^{t+1*}(c)| = q_c$ and $\mathcal{L}^{t+1}(c) = \varnothing$. In both cases, step $t+1$(c) implies that $\mu^{t+1}(c) = \mu^{t+1*}(c)$. Step $t+1$(d) then implies that the procedure will stop at step $t+1$.

□

Lemma 2

Proof. Consider some history $h \in H$. Given other players' strategies σ_{-s}, the history that results from the strategy profile (σ_s, σ_{-s}) consists, as described in the definition of the IDAM mechanism, of a series of periods in which each student has either only the action $\diamondsuit$ or some menu of options $\psi^t(s)$ to choose from. Therefore, given our strategy profile and student s, we can write down a list of pairs of menus of given to student s and her choice.

For example, suppose that the set of colleges is $C = \{c_1, c_2, c_3, c_4\}$. A possible list could be the following:

$$\left(\left(\{c_1, c_2, c_3, c_4, s\}, c_2\right)_{t=1}, \left(\varnothing, \diamondsuit\right)_{t=2}, \left(\{c_1, c_3, s\}, c_3\right)_{t=3}, \left(\varnothing, \diamondsuit\right)_{t=4=T}\right)$$

That is, in the first step the student was offered the entire list of colleges and chose c_2. In the second step, she was not offered a menu and therefore performed the continuation action $\diamondsuit$. In the third step, the student was offered colleges c_1 and c_3. Finally, during the fourth and final step, no menu was offered. Notice that, even if we don't know the strategy that was followed by the student, it would be precisely the sequence of actions taken by a student following a straightforward strategy for the preferences $c_2 P_s c_4 P_s c_3 P_s c_1 P_s s$. In fact, there is a class of preferences that are consistent with that list of pairs.

In general, say that the sequence of menus offered and actions chosen for a student s up to history h are as follows:

$$\left(\left(\psi^1, a^1\right), \left(\psi^2, a^2\right), \ldots, \left(\psi^t, a^t\right)\right)$$

For simplicity, and without any loss of generality, assume that the sequence above has removed from the list the pairs $(\varnothing, \diamondsuit)$. Because of lemma 1 and the definition of $\psi^t(s)$ in the description of the IDAM mechanism, if $a^t = c$, for all $t' > t$, $c \notin \psi^{t'}$, and therefore $a^i = a^j \implies i = j$, that is, there is no repetition of choices in a^i, $i = 1, \ldots, t$. Denote $\psi^i_- \equiv \psi^i \backslash \bigcup_{j=i}^t a^j$. We will show that this sequence could have been generated by a straightforward strategy of a student with a preference relation in the following class of preferences[17]:

$$S \backslash \psi^1 \ R^*_s \ a^1 \ P^*_s \ \psi^1_- \backslash \psi^2_- \ R^*_s \ a^2 \ P^*_s \ \psi^2_- \backslash \psi^3_- \ R^*_s \cdots R^*_s \ a^t \ P^*_s \ \psi^t_-$$

The notation above includes a class of strict preferences because some of its elements consists of sets of colleges. Any strict preference derived from some ordering over the elements of each of those sets belongs to the class of preferences that we are referring to. We will refer by P^*_s to some arbitrary element of those preferences. It is easy to see that each preference in that class is complete over the set of colleges and that no college appears more than once, since $\psi^t_- \subsetneq \psi^{t-1}_- \subsetneq \cdots \subsetneq \psi^1_- \subsetneq S$ and $a^i \notin \psi^j_-$ for all i, j.

Now, take some of the menus that were offered, ψ^i. We will now show that for for all $a \in \psi^i$ where $a \neq a^i$, $a^i P^*_s a$. For that, it suffices to show that:

$$a \in \bigcup_{j=i+1}^{t} a_j \cup \bigcup_{j=i}^{t-1} \psi^j_- \backslash \psi^{j+1}_- \cup \psi^t_-$$

That is, we will show that a must be at some element to the right of a^i in the definition of P^*_s. Since $a \neq a^i$, this is equivalent to:

$$a \in \bigcup_{j=i}^{t} a_j \cup \bigcup_{j=i}^{t-1} \psi^j_- \backslash \psi^{j+1}_- \cup \psi^t_-$$

Since we defined $\psi^i_- \equiv \psi^i \backslash \bigcup_{j=i}^t a^j$, we can rewrite the condition as:

$$a \in \underbrace{\psi^i \backslash \psi^i_-}_{(i)} \cup \underbrace{\bigcup_{j=i}^{t-1} \psi^j_- \backslash \psi^{j+1}_-}_{(ii)} \cup \underbrace{\psi^t_-}_{(iii)}$$

Suppose not. Then a cannot be in (i), (ii) and (iii). By (i), it must be that $a \notin \psi^i \backslash \psi^i_-$. Since $a \in \psi^i$, that implies $a \in \psi^i_-$. By (ii), since $a \notin \psi^i_- \backslash \psi^{i+1}_-$, it must then be that $a \in \psi^{i+1}_-$. This reasoning can be repeated until finding that it must be that $a \in \psi^t_-$. But that is (iii), which leads to a contradiction.

[17] Note that this class of preferences doesn't necessarily include all the preferences that are compatible with the choices made.

We have, therefore, that given σ_{-s}, the sequence $\left(\left(\psi^1, a^1\right), \left(\psi^2, a^2\right), \ldots, \left(\psi^t, a^t\right)\right)$ is consistent with student s having a preference over colleges P_s^* and following a straightforward strategy up to step t, since for all $a \in \psi^i$ where $a \neq a^i$, $a^i P_s^* a$. If we follow the same exercise for every student, we may construct a preference profile $P^* = \left(P_{s_1}^*, \ldots, P_{s_n}^*\right)$ where the students following straightforward strategies with respect to P^* will lead to history h. □

Theorem 1

Proof. By proposition 1, for any realization of z and P, the outcome of the strategy profile σ^* is μ^S, the student-optimal stable matching with respect to the preference profile P and college priorities z. By lemma 3, if any student s uses some deviation strategy σ_s', each realization of z and P will lead to the student-optimal stable matching for a profile $\left(P_s^*\left(P, z\right), P_{-s}, z\right)$, where $P_s^*\left(P, z\right)$ is any preference profile that could generate the history that results from the strategy profile $\left(\sigma_s'\left(\cdot|z_s, P_s\right), \sigma_{-s}^*\left(\cdot|z_{-s}, P_{-s}\right)\right)$. But Roth [1984] shows that the outcome of the student-optimal stable matching for the profile (P, z) is weakly preferred by s to that for $\left(P_s^*\left(P, z\right), P_{-s}, z\right)$. Therefore, for any realizations of z and P, student s obtains an outcome weakly better by following the straightforward strategy, given that others are following it. As a consequence, the lottery induced by the strategy profile σ^* stochastically dominates the one induced by $\left(\sigma_s'\left(\cdot|z_s, P_s\right), \sigma_{-s}^*\left(\cdot|z_{-s}, P_{-s}\right)\right)$ for player s.

Given the definition of OPBE, we still need to show that following the straightforward strategy stochastically dominates any deviation strategy at subgames that follows a player's deviation from the straightforward strategy. In other words, supposing that a player didn't follow the straighforward strategy up to period t, we need to show that following the straightforward strategy stochastically dominates any other continuation strategy, assuming that the other students follow that strategy. To see that this is true, it suffices to make two observations:

- Starting from period t, a student s's strategy is only relevant at that subgame from the moment that she is requested to make some choice at some period $t' \geqslant t$.
- At period t', from the perspective of that student, the induced subgame is indistinguishable from the IDAM mechanism that starts with students being unacceptable to schools that are not reachable anymore for them at period t'.

Since the stochastic dominance result above doesn't depend on whether a students is acceptable or not to different schools, it follows that the result also holds at subgames resulting from deviation strategies.

□

Proposition 2

Proof. Consider the set of students $S = \{s_1, s_2, s_3\}$ and of colleges $C = \{c_1, c_2, c_3\}$, each with capacity $q_i = 1$. Student s_1, who will be the player to who we will show no dominant strategy exists, has preferences $c_1 P_{s_1} c_2 P_{s_1} c_3$, and students' exam grades at those colleges are as follows:

	c_1	c_2	c_3
s_1	100	100	100
s_2	200	200	200
s_3	300	300	300

Suppose now that, conditional on the realized preferences and grades of student s_1, student s_3 follows a straightforward strategy with respect to the preference $c_3 P^3 c_2 P^3 c_1$. Notice that we are not stating those are student s_3's preferences. We are simply assuming that she follows the *straightforward strategy with respect to* P^3. Next we will consider two strategies for student s_2 and show that no strategy is a common best response for these two possibilities.

Scenario 1

Suppose that student s_2's strategy is the following: in $t = 1$, choose c_3. If at some later point s_2 is asked again to make a choice, she will choose the college with the highest cutoff value at that period among the options available. In case of ties, she will choose the college with the lowest index number (for example, the index number of c_2 is 2). We will show that, given s_2 and s_3's strategies, the unique best response involves first choosing c_2. The sequence of steps will be as follows:

Step 1: Student s_1 applies to c_2. Students s_2 and s_3 apply to c_3. Student s_2 is rejected. Cuttoffs $\left(\zeta^1_{c_1}, \zeta^1_{c_2}, \zeta^1_{c_3}\right)$ are $(0, 100, 300)$.

Step 2: Since $\zeta^1_{c_2}$ is the highest cutoff among the colleges offered to s_2, student s_2 applies to c_2. Student s_1 is rejected. Cuttoffs $\left(\zeta^2_{c_1}, \zeta^2_{c_2}, \zeta^2_{c_3}\right)$ are $(0, 200, 300)$.

Step 3: Student s_1 is left with two options: choose c_1 or s. If she chooses s she will remain unmatched. If she applies to c_1, she will be accepted. Final cuttoffs $\left(\zeta^3_{c_1}, \zeta^3_{c_2}, \zeta^3_{c_3}\right)$ would then be $(100, 200, 300)$ and the outcome would be the matching μ' as follows:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

Student s_1, therefore, can be matched to her most preferred college by first choosing c_2. We now show that by choosing first c_1 or c_3, s_1 will always me matched to a strictly inferior college. First, let her choose c_1 first:

Step 1: Student s_1 applies to c_1. Students s_2 and s_3 apply to c_3. Student s_2 is rejected. Cuttoffs $\left(\zeta^1_{c_1}, \zeta^1_{c_2}, \zeta^1_{c_3}\right)$ are $(100, 0, 300)$.

Step 2: Since $\zeta^1_{c_1}$ is the highest cutoff among the colleges offered to s_2, student s_2 applies to c_1. Student s_1 is rejected. Cuttoffs $\left(\zeta^2_{c_1}, \zeta^2_{c_2}, \zeta^2_{c_3}\right)$ are $(200, 0, 300)$.

Step 3: Student s_1 is left with two options: choose c_2 or s. If she chooses s she will remain unmatched. If she applies to c_1, she will be accepted. Final cuttoffs $\left(\zeta^3_{c_1}, \zeta^3_{c_2}, \zeta^3_{c_3}\right)$ would then be $(200, 100, 300)$ and the outcome would be the matching μ' as follows:

$$\mu' = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & s_3 \end{pmatrix}$$

If s_1 chooses c_3 first instead, the following will happen:

Step 1: Students s_1, s_2 and s_3 apply to c_3. Students s_1 and s_2 are rejected. Cuttoffs $\left(\zeta^1_{c_1}, \zeta^1_{c_2}, \zeta^1_{c_3}\right)$ are $(0, 0, 300)$.

Step 2: Following her strategy and the fact that college c_1's index is lower than c_2, student s_2 applies to c_1. Student s_1 has three options: choose also c_1 and therefore be rejected and left to choose between c_2 and s in period $t = 2$, choose c_2 or choose s. In all cases she will either end up remaining unmatched or matched to c_2.

Scenario 2

Now suppose that student s_2 follows a similar strategy to scenario 1, but where instead of applying to c_3 and then to the college with the highest cutoff value, she will apply to the college with the *lowest* cutoff value, once again breaking ties based on the index of the college[18]. Following an exercise similar to the one above, it is easy to see that student s_1's strategies that involve choosing first c_2 or c_3 will lead him to either be unmatched or be matched to c_2, while choosing c_1 will match him to c_1, her most preferred college.

Since every best response strategy under scenario 1 is dominated by different strategies in scenario 2, we have shown that a student may not have a weakly dominant strategy at the game induced by the IDAM mechanism. □

[18] Although the strategies used in this proof for student s_2 may seem very arbitrary, they can be rationalized by two simple stories. Student s_2's strategy in scenario 1 is consistent with a student who knows that her top choice is c_3 but that has some uncertainty about which one among c_1 and c_2 is her second choice, and sees the cut-off grade as an indication of how competitive is acceptance at those colleges and therefore the perceived quality of those. The strategy in scenario 2 could be rationalized by a student who once again knows that her top choice is c_3 but that would prefer otherwise to go with a college with low-achieving peers, and uses the low cut-off as an indication of that fact.